

On estimation and the occurrence of boundary solutions in incomplete tables

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ABSTRACT. A useful and efficient method for analyzing incomplete tables is to model the missing data mechanism using log-linear models. Iterative algorithms are available in literature to obtain estimates of expected cell counts in such tables. In this paper, we discuss log-linear parametrization and estimation in three-way and n -dimensional tables with missing data. We provide simple closed form estimates of expected counts and parameters under the various models, which reduces computations substantially. Further, we also discuss the issue of boundary solutions and their various forms for the above tables. We provide sufficient conditions for the occurrence of boundary solutions in nonignorable nonresponse models for arbitrary three-way and n -dimensional incomplete tables. These conditions involve only the observed counts, and do not require solving likelihood equations or using the EM algorithm. A real-life dataset is analyzed to illustrate our results for both the estimation and the occurrence of boundary solutions in incomplete tables.

1. INTRODUCTION

The analysis of contingency tables with missing data, also called incomplete tables, is of practical interest in statistics. There are two types of counts in such tables: (i) fully observed counts and (ii) partially classified margins (nonresponses). A systematic study of missing data involves three types of missingness mechanisms proposed in the literature (see Little and Rubin (2002)) : missing completely at random (MCAR), missing at random (MAR) and not missing at random (NMAR). A mechanism is said to be MCAR when the probability of an observation being missing is independent of both observed and unobserved data, MAR if conditional on the observed data, the probability is independent of unobserved data, and NMAR if the probability depends only on unobserved data. For likelihood inference, nonresponses are classified as either ignorable (when the missing data mechanism is MAR or MCAR) or nonignorable (when the missing data mechanism is NMAR).

Log-linear models have generally been used to analyze missing data mechanisms in incomplete contingency tables (see Baker and Laird (1988); Baker *et al.* (1992); Smith *et al.* (1999); Clarke (2002); Clarke and Smith (2004, 2005)). In this paper, we extend the log-linear parametrization and the estimation technique proposed by Baker *et al.* (1992) for an $I \times J \times 2 \times 2$ table to three-dimensional and n -dimensional incomplete tables in general. We consider all possible cases when data on one or more of the variables are missing. Simple closed-form algebraic formulae are provided for estimates of expected cell counts and parameters under various models in the above tables. This eliminates the need for iterative

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procedures like the EM algorithm (see Dempster *et al.* (1977)) and thereby reduces the computational burden. Note that the problem of boundary solutions occurs in such models under the NMAR mechanism while using ML estimation. For a $2 \times 2 \times 2$ incomplete contingency table, Baker and Laird (1988) proposed a condition for the occurrence of boundary solutions in NMAR models. For an $I \times J \times 2 \times 2$ table, Baker *et al.* (1992) showed that boundary solutions under NMAR models may occur when certain systems of likelihood equations are solved to obtain the parameter estimates. The boundary solution problem was geometrically described by Smith *et al.* (1999) and Clarke (2002). Clarke and Smith (2005) described properties of maximum likelihood estimators for NMAR models when boundary solutions occur. Recently, Park *et al.* (2014) provided a sufficient condition for the occurrence of boundary solutions in NMAR models when data on both variables are missing in square two-way incomplete tables.

In this paper, sufficient conditions are established for the same in several identifiable NMAR models for $I \times J \times K \times 2$, $I \times J \times K \times 2 \times 2$, $I \times J \times K \times 2 \times 2 \times 2$ and arbitrary n -dimensional incomplete tables. These conditions require only the fully observed and partially classified cell counts. Hence, they are easily verifiable and eliminate the need for using the EM algorithm or solving likelihood equations. The remaining part of the paper is organized as follows. In Section 2, we provide log-linear parametrizations and discuss estimation methods for $I \times J \times K \times 2$, $I \times J \times K \times 2 \times 2$, $I \times J \times K \times 2 \times 2 \times 2$ incomplete tables. In Section 3, various identifiable NMAR models for such tables are considered. Section 4 deals with boundary solutions, their various forms and sufficient conditions for their occurrence under the NMAR models in each of the above tables. Section 5 extends the discussions and results in Sections 2-4 to arbitrary n -dimensional incomplete tables. A real-life data analysis example is presented in Section 6 to illustrate the results in Sections 2 and 4. Section 7 provides some concluding remarks.

2. LOG-LINEAR PARAMETRIZATION FOR 3-DIMENSIONAL INCOMPLETE TABLES

Baker *et al.* (1992) suggested nine identifiable log-linear models for studying missing data mechanisms in an $I \times J \times 2 \times 2$ incomplete table. In this section, we discuss such hierarchical log-linear models for three-way contingency tables where data on one, two or all variables may be missing. It is assumed that partially classified (supplementary) margins are positive. Also, null fully observed counts do not appear in the likelihood function and hence the likelihood ratio statistic. Now suppose Y_1 , Y_2 and Y_3 are three categorical variables with I , J and K levels respectively. Then we have the following cases.

2.1. One of the variables is missing. Let Y_1 be missing and R denote the missing indicator for Y_1 such that $R = 1$ if Y_1 is observed and $R = 2$ otherwise. Then for Y_1 , Y_2 , Y_3 and R , we have an $I \times J \times K \times 2$ table with cell counts $\mathbf{y} = \{y_{ijkx}\}$ where $1 \leq i \leq I$, $1 \leq j \leq J$, $1 \leq k \leq K$ and $x = 1, 2$. The vector of observed counts is $\mathbf{y}_{\text{obs}} = (\{y_{ijk1}\}, \{y_{+jk2}\})$, where $\{y_{ijk1}\}$ are the fully observed counts and $\{y_{+jk2}\}$ are the supplementary margins. Note that ‘+’ denotes summation over levels of the corresponding variable. Let $\pi = \{\pi_{ijkx}\}$ be the vector of cell probabilities, $\mu = \{\mu_{ijkx}\}$ be the vector of expected counts and $N = \sum_{i,j,k,x} y_{ijkx}$ be the total cell count. For $I = J = K = 2$, the $2 \times 2 \times 2 \times 2$ incomplete table is given below.

Table 1. $2 \times 2 \times 2 \times 2$ Incomplete Table.

			$Y_3 = 1$	$Y_3 = 2$
$R = 1$	$Y_1 = 1$	$Y_2 = 1$	y_{1111}	y_{1121}
		$Y_2 = 2$	y_{1211}	y_{1221}
	$Y_1 = 2$	$Y_2 = 1$	y_{2111}	y_{2121}
		$Y_2 = 2$	y_{2211}	y_{2221}
$R = 2$	Missing	$Y_2 = 1$	y_{+112}	y_{+122}
		$Y_2 = 2$	y_{+212}	y_{+222}

The log-linear model (with no three-way or four-way interactions) is then given by

$$(1) \quad \begin{aligned} \log \mu_{ijkx} = & \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_3}(k) + \lambda_R(x) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_2 Y_3}(j, k) \\ & + \lambda_{Y_1 R}(i, x) + \lambda_{Y_2 R}(j, x) + \lambda_{Y_3 R}(k, x). \end{aligned}$$

Each log-linear parameter in (1) satisfies the constraint that the sum over each of its arguments is 0, for example, $\sum_i \lambda_{Y_1 Y_3}(i, k) = \sum_k \lambda_{Y_1 Y_3}(i, k) = 0$. Define $a_{ijk} = \frac{P(R=2|Y_1=i, Y_2=j, Y_3=k)}{P(R=1|Y_1=i, Y_2=j, Y_3=k)} = \frac{\pi_{ijk2}}{\pi_{ijk1}} = \frac{\mu_{ijk2}}{\mu_{ijk1}}$, which describes the missing data mechanism of Y_1 . It is the odds of Y_1 being missing. Let $m_{ijk1} = \mu_{ijk1} \Rightarrow \mu_{ijk2} = a_{ijk} m_{ijk1}$. Also, $\sum_{i,j,k} m_{ijk1}(1 + a_{ijk}) = N$ and the joint probability $\pi_{ijk.} = m_{ijk1}(1 + a_{ijk})/N$, from which the marginals may be derived. Note that under (1), $a_{ijk} = \exp[-2\{\lambda_R(1) + \lambda_{Y_1 R}(i, 1) + \lambda_{Y_2 R}(j, 1) + \lambda_{Y_3 R}(k, 1)\}]$. Denote a_{ijk} by $\alpha_{i..}$ or $\alpha_{.j.}$ or $\alpha_{..k}$ or $\alpha_{...}$ if it depends on i or j or k or none, respectively.

Definition 2.1. The missing mechanism of Y_1 under (1) is NMAR if $a_{ijk} = \alpha_{i..}$, MAR if $a_{ijk} = \alpha_{.j.}$ or $\alpha_{..k}$ and MCAR if $a_{ijk} = \alpha_{...}$.

Under Poisson sampling for observed cell counts, the log-likelihood of μ is

$$(2) \quad l(\mu; \mathbf{y}_{\text{obs}}) = \sum_{i,j,k} y_{ijk1} \log \mu_{ijk1} + \sum_{j,k} y_{+jk2} \log \mu_{+jk2} - \sum_{i,j,k,x} \mu_{ijkx} + \Delta,$$

where Δ is some constant. The various missing data models and the MLE's under them are given as follows :

1. $\alpha_{i..}$ (NMAR for Y_1).

We have $\hat{m}_{ijk1} = y_{ijk1}$ and $\hat{\alpha}_{i..}$ satisfies $\sum_i \hat{m}_{ijk1} \hat{\alpha}_{i..} = y_{+jk2} \forall 1 \leq j \leq J, 1 \leq k \leq K$.

2. $\alpha_{.j.}$ (MAR for Y_1).

We have $\hat{m}_{ijk1} = \frac{y_{ijk1} y_{+jk+} y_{++j+1}}{y_{+jk1} y_{++j+}}$ and $\hat{\alpha}_{.j.} = \frac{y_{++j+2}}{y_{++j+1}}$.

3. $\alpha_{..k}$ (MAR for Y_1).

We have $\hat{m}_{ijk1} = \frac{y_{ijk1} y_{+jk+} y_{++k+1}}{y_{+jk1} y_{++k+}}$ and $\hat{\alpha}_{..k} = \frac{y_{++k+2}}{y_{++k+1}}$.

4. $\alpha_{...}$ (MCAR for Y_1).

We have $\hat{m}_{ijk1} = \frac{y_{ijk1} y_{+jk+} y_{+++1}}{y_{+jk1} y_{+++}}$ and $\hat{\alpha}_{...} = \frac{y_{+++2}}{y_{+++1}}$.

Boundary solutions occur if $\hat{\alpha}_{i..} \leq 0$ for at least one and at most $(I - 1)$ values of Y_1 (see Baker *et al.* (1992)). If any $\hat{\alpha}_{i..} < 0$, then boundary estimates are obtained by setting $\hat{\alpha}_{i..} = 0$ in (2). For example, if $Y_1 = 1, 2$ and $\hat{\alpha}_{1..} = 0$ under model 1, then the MLE's are

$$\hat{\alpha}_{2..} = \frac{y_{+++2}}{y_{+++1}}, \quad \hat{m}_{1jk1} = y_{1jk1}, \quad \hat{m}_{2jk1} = \frac{(y_{2jk1} + y_{+jk2}) y_{2+++1}}{y_{+++2}}.$$

Consider the following hypotheses : H_0 - the proposed model (among models 1 to 4 mentioned above) fits the data, and H_1 - the perfect fit model fits the data. Let L_0 and L_1 denote the maximized log-likelihood functions under the proposed and perfect fit models respectively. Then the likelihood ratio statistic for testing H_0 against H_1 is given by

$$G^2 = -2(L_0 - L_1)$$

$$(3) = -2 \left[\sum_{i,j,k} y_{ijk1} \ln \left(\frac{\hat{m}_{ijk1}}{y_{ijk1}} \right) + \sum_{j,k} y_{+jk2} \ln \left(\frac{\sum_i \hat{m}_{ijk1} \hat{a}_{ijk}}{y_{+jk2}} \right) - \sum_{i,j,k} \hat{m}_{ijk1} (1 + \hat{a}_{ijk}) + N \right].$$

Note that G^2 follows χ^2_ν asymptotically, where $\nu = (I + 1)JK$ (Number of observed counts) – Number of free estimable parameters under the proposed model.

2.2. Two of the variables are missing. Suppose Y_1 and Y_2 are missing and for $i = 1, 2$, let R_i denote the missing indicator for Y_i such that $R_i = 1$ if Y_i is observed and $R_i = 2$ otherwise. Then for Y_1, Y_2, Y_3, R_1 and R_2 , we have an $I \times J \times K \times 2 \times 2$ table with cell counts $\mathbf{y} = \{y_{ijkxs}\}$, where $1 \leq i \leq I, 1 \leq j \leq J, 1 \leq k \leq K$ and $x, s = 1, 2$. The vector of observed counts is $\mathbf{y}_{\text{obs}} = (\{y_{ijk11}\}, \{y_{+jk21}\}, \{y_{i+k12}\}, \{y_{++k22}\})$. Let $\pi = \{\pi_{ijkxs}\}$ be the vector of cell probabilities, $\mu = \{\mu_{ijkxs}\}$ be the vector of expected counts and N be the total cell count. For $I = J = K = 2$, the $2 \times 2 \times 2 \times 2 \times 2$ incomplete table is given below.

Table 2. $2 \times 2 \times 2 \times 2 \times 2$ Incomplete Table.

				$Y_3 = 1$	$Y_3 = 2$
$R_1 = 1$	$Y_1 = 1$	$R_2 = 1$	$Y_2 = 1$	y_{11111}	y_{11211}
			$Y_2 = 2$	y_{12111}	y_{12211}
		$R_2 = 2$	Missing	y_{1+112}	y_{1+212}
	$Y_1 = 2$	$R_2 = 1$	$Y_2 = 1$	y_{21111}	y_{21211}
			$Y_2 = 2$	y_{22111}	y_{22211}
		$R_2 = 2$	Missing	y_{2+112}	y_{2+212}
$R_1 = 2$	Missing	$R_2 = 1$	$Y_2 = 1$	y_{+1121}	y_{+1221}
			$Y_2 = 2$	y_{+2121}	y_{+2221}
		$R_2 = 2$	Missing	y_{++122}	y_{++222}

The log-linear model (with no three-way or higher order interactions) is given by

$$(4) \quad \begin{aligned} \log \mu_{ijkxs} = & \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_3}(k) + \lambda_{R_1}(x) + \lambda_{R_2}(s) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_3}(i, k) \\ & + \lambda_{Y_2 Y_3}(j, k) + \lambda_{Y_1 R_1}(i, x) + \lambda_{Y_2 R_1}(j, x) + \lambda_{Y_3 R_1}(k, x) \\ & + \lambda_{Y_1 R_2}(i, s) + \lambda_{Y_2 R_2}(j, s) + \lambda_{Y_3 R_2}(k, s) + \lambda_{R_1 R_2}(x, s). \end{aligned}$$

Each log-linear parameter in (4) satisfies the constraint that the sum over each of its arguments is 0. Define the following quantities

$$a_{ijk} = \frac{P(R_1 = 2, R_2 = 1 \mid Y_1 = i, Y_2 = j, Y_3 = k)}{P(R_1 = 1, R_2 = 1 \mid Y_1 = i, Y_2 = j, Y_3 = k)} = \frac{\pi_{ijk21}}{\pi_{ijk11}} = \frac{\mu_{ijk21}}{\mu_{ijk11}},$$

$$b_{ijk} = \frac{P(R_1 = 1, R_2 = 2 \mid Y_1 = i, Y_2 = j, Y_3 = k)}{P(R_1 = 1, R_2 = 1 \mid Y_1 = i, Y_2 = j, Y_3 = k)} = \frac{\pi_{ijk12}}{\pi_{ijk11}} = \frac{\mu_{ijk12}}{\mu_{ijk11}}.$$

Then a_{ijk} and b_{ijk} describe the missing data mechanisms of Y_1 and Y_2 respectively. Note that a_{ijk} is the conditional odds of Y_1 being missing given Y_2 is observed, while b_{ijk} is the conditional odds of Y_2 being missing given Y_1 is observed. The odds ratio between R_1 and R_2 is

$$\begin{aligned}\theta &= \frac{P(R_1 = 1, R_2 = 1 \mid Y_1 = i, Y_2 = j, Y_3 = k)P(R_1 = 2, R_2 = 2 \mid Y_1 = i, Y_2 = j, Y_3 = k)}{P(R_1 = 1, R_2 = 2 \mid Y_1 = i, Y_2 = j, Y_3 = k)P(R_1 = 2, R_2 = 1 \mid Y_1 = i, Y_2 = j, Y_3 = k)} \\ &= \frac{\pi_{ijk11}\pi_{ijk22}}{\pi_{ijk12}\pi_{ijk21}} = \frac{\mu_{ijk11}\mu_{ijk22}}{\mu_{ijk12}\mu_{ijk21}}.\end{aligned}$$

If $\theta = 1$, then the missingness patterns of Y_1 and Y_2 , that is, R_1 and R_2 are independent. Let $m_{ijk11} = \mu_{ijk11}$. Then $\mu_{ijk21} = a_{ijk}m_{ijk11}$, $\mu_{ijk12} = b_{ijk}m_{ijk11}$, $\mu_{ijk22} = m_{ijk11}a_{ijk}b_{ijk}\theta$ and $N = \sum_{i,j,k} m_{ijk11}(1 + a_{ijk} + b_{ijk} + a_{ijk}b_{ijk}\theta)$. The joint probability is $\pi_{ijk..} = m_{ijk11}(1 + a_{ijk} + b_{ijk} + a_{ijk}b_{ijk}\theta)/N$, from which the marginals can be obtained. The conditional probability of Y_1 being missing given that Y_2 is observed is

$$\phi_{1|2}(i, j, k) = P(R_1 = 2 \mid R_2 = 1, Y_1 = i, Y_2 = j, Y_3 = k) = \frac{a_{ijk}}{1 + a_{ijk}}.$$

Similarly, the conditional probability of Y_2 being missing given that Y_1 is observed is

$$\phi_{2|1}(i, j, k) = P(R_2 = 2 \mid R_1 = 1, Y_1 = i, Y_2 = j, Y_3 = k) = \frac{b_{ijk}}{1 + b_{ijk}}.$$

Under (4), $a_{ijk} = \exp[-2\{\lambda_{R_1}(1) + \lambda_{Y_1 R_1}(i, 1) + \lambda_{Y_2 R_1}(j, 1) + \lambda_{Y_3 R_1}(k, 1) + \lambda_{R_1 R_2}(1, 1)\}]$, $b_{ijk} = \exp[-2\{\lambda_{R_2}(1) + \lambda_{Y_1 R_2}(i, 1) + \lambda_{Y_2 R_2}(j, 1) + \lambda_{Y_3 R_2}(k, 1) + \lambda_{R_1 R_2}(1, 1)\}]$ and $\theta = \exp[4\lambda_{R_1 R_2}(1, 1)]$. If each of a_{ijk} and b_{ijk} depends on one of i, j, k or none, then let $a_{ijk} = (\alpha_{i..}, \alpha_{.j.}, \alpha_{..k}, \alpha_{...})$ and $b_{ijk} = (\beta_{i..}, \beta_{.j.}, \beta_{..k}, \beta_{...})$.

Definition 2.2. The missing mechanism of Y_1 under (4) is NMAR if $a_{ijk} = \alpha_{i..}$, MAR if $a_{ijk} = \alpha_{.j.}$ or $\alpha_{..k}$ and MCAR if $a_{ijk} = \alpha_{...}$, respectively. Similarly, the missing mechanism of Y_2 is NMAR if $b_{ijk} = \beta_{.j.}$, MAR if $b_{ijk} = \beta_{i..}$ or $\beta_{..k}$ and MCAR if $b_{ijk} = \beta_{...}$.

There are 16 identifiable models in this case. Under Poisson sampling, the log-likelihood kernel of μ is

$$\begin{aligned}l(\mu; \mathbf{y}_{\text{obs}}) &= \sum_{i,j,k} y_{ijk11} \log \mu_{ijk11} + \sum_{j,k} y_{+jk21} \log \mu_{+jk21} + \sum_{i,k} y_{i+k12} \log \mu_{i+k12} \\ (5) \quad &+ \sum_k y_{++k22} \log \mu_{++k22} - \sum_{i,j,k,x,s} \mu_{ijkxs}.\end{aligned}$$

The various models and the MLE's under them are given as follows.

1. $(\alpha_{...}, \beta_{...})$ (MCAR for both Y_1 and Y_2).

The MLE's are

$$\hat{\alpha}_{...} = \frac{y_{+++21}}{y_{+++11}}, \quad \hat{\beta}_{...} = \frac{y_{+++12}}{y_{+++11}}, \quad \hat{\theta} = \frac{y_{+++11}y_{+++22}}{y_{+++12}y_{+++21}},$$

while the iterates of \hat{m}_{ijk11} are

$$\hat{m}_{ijk11}^{(0)} = y_{ijk11}, \quad \hat{m}_{ijk11}^{(t+1)} = \frac{y_{+++11} \left(y_{ijk11} + \frac{y_{i+k12}}{\hat{m}_{i+k12}^{(t)}} \cdot \hat{m}_{ijk11}^{(t)} + \frac{y_{+jk21}}{\hat{m}_{+jk21}^{(t)}} \cdot \hat{m}_{ijk11}^{(t)} \right)}{y_{++++1} + y_{++++2}}.$$

2. $(\alpha_{...}, \beta_{i..})$ (MCAR for Y_1 , MAR for Y_2).

The MLE's are

$$\hat{\alpha}_{...} = \frac{y_{+++21}}{y_{+++11}}, \hat{\beta}_{i..} = \frac{y_{i++12}}{\hat{m}_{i++11}}, \hat{\theta} = \frac{y_{+++11}y_{+++22}}{y_{+++12}y_{+++21}}, \hat{m}_{ijk11} = \frac{y_{ijk11}y_{+++11}y_{+jk+1}}{y_{++++1}y_{+jk11}}.$$

3. $(\alpha_{...}, \beta_{.j.})$ (MCAR for Y_1 , NMAR for Y_2).

The MLE's are

$$\hat{\alpha}_{...} = \frac{y_{+++21}}{y_{+++11}}, \hat{\theta} = \frac{y_{+++11}y_{+++22}}{y_{+++12}y_{+++21}}, \hat{m}_{ijk11} = \frac{y_{ijk11}y_{+++11}y_{+jk+1}}{y_{++++1}y_{+jk11}}.$$

Also, $\hat{\beta}_{.j.}$ satisfies $\sum_j \hat{m}_{ijk11} \hat{\beta}_{.j.} = y_{i+k12}$.

4. $(\alpha_{...}, \beta_{..k})$ (MCAR for Y_1 , MAR for Y_2).

The MLE's are

$$\hat{\alpha}_{...} = \frac{y_{+++21}}{y_{+++11}}, \hat{\beta}_{..k} = \frac{y_{++k12}}{\hat{m}_{++k11}}, \hat{\theta} = \frac{y_{+++11}y_{+++22}}{y_{+++12}y_{+++21}}, \hat{m}_{ijk11} = \frac{y_{ijk11}y_{+++11}y_{+jk+1}}{y_{++++1}y_{+jk11}}.$$

5. $(\alpha_{i..}, \beta_{...})$ (NMAR for Y_1 , MCAR for Y_2).

The MLE's are

$$\hat{\beta}_{...} = \frac{y_{+++12}}{y_{+++11}}, \hat{\theta} = \frac{y_{+++11}y_{+++22}}{y_{+++12}y_{+++21}}, \hat{m}_{ijk11} = \frac{y_{ijk11}y_{+++11}y_{i+k1+}}{y_{++++1}y_{i+k11}}.$$

Also, $\hat{\alpha}_{i..}$ satisfies $\sum_i \hat{m}_{ijk11} \hat{\alpha}_{i..} = y_{+jk21}$.

6. $(\alpha_{i..}, \beta_{i..})$ (NMAR for Y_1 , MAR for Y_2).

The MLE's are

$$\hat{m}_{ijk11} = y_{ijk11}, \hat{\beta}_{i..} = \frac{y_{i++12}}{y_{i++11}}, \hat{\theta} = \frac{y_{+++22}}{\sum_i y_{i++12} \hat{\alpha}_{i..}},$$

where $\hat{\alpha}_{i..}$ satisfies $\sum_i \hat{m}_{ijk11} \hat{\alpha}_{i..} = y_{+jk21}$.

7. $(\alpha_{i..}, \beta_{.j.})$ (NMAR for both Y_1 and Y_2).

The MLE's are

$$\hat{m}_{ijk11} = y_{ijk11}, \hat{\theta} = \frac{y_{+++22}}{\sum_{i,j} y_{ij+11} \hat{\alpha}_{i..} \hat{\beta}_{.j.}},$$

where $\hat{\alpha}_{i..}$ and $\hat{\beta}_{.j.}$ satisfy $\sum_i \hat{m}_{ijk11} \hat{\alpha}_{i..} = y_{+jk21}$ and $\sum_j \hat{m}_{ijk11} \hat{\beta}_{.j.} = y_{i+k12}$ respectively.

8. $(\alpha_{i..}, \beta_{..k})$ (NMAR for Y_1 , MAR for Y_2).

The MLE's are

$$\hat{m}_{ijk11} = y_{ijk11}, \hat{\beta}_{..k} = \frac{y_{++k12}}{y_{++k11}}, \hat{\theta} = \frac{y_{+++22}}{\sum_{i,k} y_{i+k11} \hat{\alpha}_{i..} \hat{\beta}_{..k}},$$

where $\hat{\alpha}_{i..}$ satisfies $\sum_i \hat{m}_{ijk11} \hat{\alpha}_{i..} = y_{+jk21}$.

9. $(\alpha_{.j.}, \beta_{...})$ (MAR for Y_1 , MCAR for Y_2).

The MLE's are

$$\hat{\alpha}_{.j.} = \frac{y_{+j+21}}{\hat{m}_{+j+11}}, \hat{\beta}_{...} = \frac{y_{+++12}}{y_{+++11}}, \hat{\theta} = \frac{y_{+++11}y_{+++22}}{y_{+++12}y_{+++21}}, \hat{m}_{ijk11} = \frac{y_{ijk11}y_{+++11}y_{i+k1+}}{y_{++++1}y_{i+k11}}.$$

10. $(\alpha_{.j.}, \beta_{i..})$ (MAR for both Y_1 and Y_2).

The MLE's are

$$\hat{m}_{ijk11} = y_{ijk11}, \hat{\alpha}_{.j.} = \frac{y_{+j+21}}{y_{+j+11}}, \hat{\beta}_{i..} = \frac{y_{i++12}}{y_{i++11}}, \hat{\theta} = \frac{y_{+++22}}{\sum_{i,j} y_{ij+11} \hat{\alpha}_{.j.} \hat{\beta}_{i..}}.$$

11. $(\alpha_{.j}, \beta_{.j})$ (MAR for Y_1 , NMAR for Y_2).

The MLE's are

$$\hat{m}_{ijk11} = y_{ijk11}, \quad \hat{\alpha}_{.j} = \frac{y_{+j+21}}{y_{+j+11}}, \quad \hat{\theta} = \frac{y_{+++22}}{\sum_j y_{+j+21} \hat{\beta}_{.j}},$$

where $\hat{\beta}_{.j}$ satisfies $\sum_j \hat{m}_{ijk11} \hat{\beta}_{.j} = y_{i+k12}$.

12. $(\alpha_{.j}, \beta_{..k})$ (MAR for both Y_1 and Y_2).

The MLE's are

$$\hat{m}_{ijk11} = y_{ijk11}, \quad \hat{\alpha}_{.j} = \frac{y_{+j+21}}{y_{+j+11}}, \quad \hat{\beta}_{..k} = \frac{y_{++k12}}{y_{++k11}}, \quad \hat{\theta} = \frac{y_{+++22}}{\sum_{j,k} y_{+jk11} \hat{\alpha}_{.j} \hat{\beta}_{..k}}.$$

13. $(\alpha_{..k}, \beta_{...})$ (MAR for Y_1 , MCAR for Y_2).

The MLE's are

$$\hat{\alpha}_{..k} = \frac{y_{++k21}}{\hat{m}_{++k11}}, \quad \hat{\beta}_{...} = \frac{y_{+++12}}{y_{+++11}}, \quad \hat{\theta} = \frac{y_{+++11} y_{+++22}}{y_{+++12} y_{+++21}}, \quad \hat{m}_{ijk11} = \frac{y_{ijk11} y_{+++11} y_{i+k1+}}{y_{+++1+} y_{i+k11}}.$$

14. $(\alpha_{..k}, \beta_{i..})$ (MAR for both Y_1 and Y_2).

The MLE's are

$$\hat{m}_{ijk11} = y_{ijk11}, \quad \hat{\alpha}_{..k} = \frac{y_{++k21}}{y_{++k11}}, \quad \hat{\beta}_{i..} = \frac{y_{i++12}}{y_{i++11}}, \quad \hat{\theta} = \frac{y_{+++22}}{\sum_{i,k} y_{i+k11} \hat{\alpha}_{..k} \hat{\beta}_{i..}}.$$

15. $(\alpha_{..k}, \beta_{.j})$ (MAR for Y_1 , NMAR for Y_2).

The MLE's are

$$\hat{m}_{ijk11} = y_{ijk11}, \quad \hat{\alpha}_{..k} = \frac{y_{++k21}}{y_{++k11}}, \quad \hat{\theta} = \frac{y_{+++22}}{\sum_{j,k} y_{+jk11} \hat{\alpha}_{..k} \hat{\beta}_{.j}},$$

where $\hat{\beta}_{.j}$ satisfies $\sum_j \hat{m}_{ijk11} \hat{\beta}_{.j} = y_{i+k12}$.

16. $(\alpha_{..k}, \beta_{..k})$ (MAR for both Y_1 and Y_2).

The MLE's are

$$\hat{m}_{ijk11} = y_{ijk11}, \quad \hat{\alpha}_{..k} = \frac{y_{++k21}}{y_{++k11}}, \quad \hat{\beta}_{..k} = \frac{y_{++k12}}{y_{++k11}}, \quad \hat{\theta} = \frac{y_{+++22}}{\sum_k y_{++k12} \hat{\alpha}_{..k} \hat{\beta}_{..k}}.$$

Note that closed-form MLE's of m_{ijk11} exist for all models except for model 1. In this case, \hat{m}_{ijk11} may be obtained using the EM algorithm. Boundary solutions occur under at least one of the following cases.

1. $\hat{\alpha}_{i..} \leq 0$ for at least one and at most $(I - 1)$ values of Y_1 ,

2. $\hat{\beta}_{.j} \leq 0$ for at least one and at most $(J - 1)$ values of Y_2 .

They occur in models in which the missing mechanism of at least one of the variables is NMAR. If any $\hat{\alpha}_{i..} < 0$ or any $\hat{\beta}_{.j} < 0$, then boundary estimates can still be obtained by setting $\hat{\alpha}_{i..} = \hat{\beta}_{.j} = 0$ in (5) for relevant models. Now suppose $Y_1, Y_2 = 1, 2$. Then we have a $2 \times 2 \times K \times 2 \times 2$ incomplete contingency table. The boundary MLE's obtained when $\hat{\alpha}_{1..} = 0$ or $\hat{\beta}_{.2} = 0$ (say) under various NMAR models are shown below.

(a) $(\alpha_{i..}, \beta_{...})$ (NMAR for Y_1 , MCAR for Y_2) :

If $\hat{\alpha}_{1..} = 0$, then the MLE's are

$$\begin{aligned}\hat{\alpha}_{2..} &= \frac{y_{+++21}y_{+++1+}}{y_{+++11}y_{2++1+}}, \quad \hat{\beta}_{...} = \frac{y_{+++12}}{y_{+++11}}, \quad \hat{\theta} = \frac{y_{+++11}y_{+++22}}{y_{+++12}y_{+++21}}, \quad \hat{m}_{1jk11} = \frac{y_{1jk11}y_{1++1+}y_{+++11}}{y_{1++11}y_{+++1+}}, \\ \hat{m}_{2jk11} &= \frac{y_{+++11}y_{2++1+}(y_{2jk11} + y_{+jk21})}{y_{+++1+}(y_{2++11} + y_{+++21})}.\end{aligned}$$

(b) $(\alpha_{i..}, \beta_{i..})$ (NMAR for Y_1 , MAR for Y_2) :

If $\hat{\alpha}_{1..} = 0$, then the MLE's are

$$\begin{aligned}\hat{\alpha}_{2..} &= \frac{y_{+++21}}{y_{2++11}}, \quad \hat{\beta}_{i..} = \frac{y_{i++12}}{y_{i++11}}, \quad \hat{\theta} = \frac{y_{2++11}y_{+++22}}{y_{2++12}y_{+++21}}, \\ \hat{m}_{1jk11} &= y_{1jk11}, \quad \hat{m}_{2jk11} = \frac{y_{2++11}(y_{2jk11} + y_{+jk21})}{y_{2++11} + y_{+++21}}.\end{aligned}$$

(c) $(\alpha_{i..}, \beta_{.j.})$ (NMAR for both Y_1 and Y_2) :

(i) If $\hat{\alpha}_{1..} = 0$, then the MLE's are

$$\hat{\alpha}_{2..} = \frac{y_{+++21}}{y_{2++11}}, \quad \hat{\theta} = \frac{y_{2++11}y_{+++22}}{y_{2++12}y_{+++21}}, \quad \hat{m}_{1jk11} = y_{1jk11}, \quad \hat{m}_{2jk11} = \frac{y_{2++11}(y_{2jk11} + y_{+jk21})}{y_{2++11} + y_{+++21}}.$$

Also, $\hat{\beta}_{.j.}$ satisfies $\sum_j \hat{m}_{ijk11} \hat{\beta}_{.j.} = y_{i+k12}$.

(ii) If $\hat{\beta}_{.2.} = 0$, then the MLE's are

$$\hat{\beta}_{.1.} = \frac{y_{+++12}}{y_{+1+11}}, \quad \hat{\theta} = \frac{y_{+1+11}y_{+++22}}{y_{+1+12}y_{+++21}}, \quad \hat{m}_{i1k11} = \frac{y_{+1+11}(y_{i1k11} + y_{i+k12})}{y_{+1+11} + y_{+++21}}, \quad \hat{m}_{i2k11} = y_{i2k11}.$$

Also, $\hat{\alpha}_{i..}$ satisfies $\sum_i \hat{m}_{ijk11} \hat{\alpha}_{i..} = y_{+jk21}$.

(d) $(\alpha_{...}, \beta_{.j.})$ (NMAR for Y_2 , MCAR for Y_1) :

If $\hat{\beta}_{.2.} = 0$, then the MLE's are

$$\begin{aligned}\hat{\beta}_{.1.} &= \frac{y_{+++12}y_{+++++1}}{y_{+++11}y_{+1++1}}, \quad \hat{\alpha}_{...} = \frac{y_{+++12}}{y_{+++11}}, \quad \hat{\theta} = \frac{y_{+++11}y_{+++22}}{y_{+++12}y_{+++21}}, \\ \hat{m}_{i1k11} &= \frac{y_{+++11}y_{+1++1}(y_{i1k11} + y_{i+k12})}{y_{+++++1}(y_{+1++11} + y_{+++12})}, \quad \hat{m}_{i2k11} = \frac{y_{i2k11}y_{+2++1}y_{+++11}}{y_{+2++11}y_{+++++1}}.\end{aligned}$$

(e) $(\alpha_{.j.}, \beta_{.j.})$ (NMAR for Y_2 , MAR for Y_1) :

If $\hat{\beta}_{.2.} = 0$, then the MLE's are

$$\begin{aligned}\hat{\beta}_{.1.} &= \frac{y_{+++12}}{y_{+1+11}}, \quad \hat{\alpha}_{.j.} = \frac{y_{+jk21}}{y_{+jk11}}, \quad \hat{\theta} = \frac{y_{+1+11}y_{+++22}}{y_{+1+12}y_{+++21}}, \\ \hat{m}_{i1k11} &= \frac{y_{+1+11}(y_{i1k11} + y_{i+k12})}{y_{+1+11} + y_{+++21}}, \quad \hat{m}_{i2k11} = y_{i2k11}.\end{aligned}$$

Next consider testing the hypotheses H_0 : the proposed model (among models 1 to 16 mentioned above) fits the data against H_1 : the perfect fit model fits the data. Let L_0 and L_1 denote the maximized log-likelihood functions under the proposed and perfect fit models respectively.

Then the likelihood ratio statistic for testing H_0 against H_1 is given by

$$\begin{aligned}
G^2 &= -2(L_0 - L_1) \\
&= -2 \left[\sum_{i,j,k} y_{ijk11} \ln \left(\frac{\hat{m}_{ijk11}}{y_{ijk11}} \right) + \sum_{j,k} y_{+jk21} \ln \left(\frac{\sum_i \hat{m}_{ijk11} \hat{a}_{ijk}}{y_{+jk21}} \right) \right. \\
&\quad + \sum_{i,k} y_{i+k12} \ln \left(\frac{\sum_j \hat{m}_{ijk11} \hat{b}_{ijk}}{y_{i+k12}} \right) + \sum_k y_{++k22} \ln \left(\frac{\sum_{i,j} \hat{m}_{ijk11} \hat{a}_{ijk} \hat{b}_{ijk} \hat{\theta}}{y_{++k22}} \right) \\
&\quad \left. - \sum_{i,j,k} \hat{m}_{ijk11} (1 + \hat{a}_{ijk} + \hat{b}_{ijk} + \hat{a}_{ijk} \hat{b}_{ijk} \hat{\theta}) + N \right].
\end{aligned}
\tag{6}$$

Note that G^2 follows χ_ν^2 asymptotically, where $\nu = (I+1)(J+1)K - \text{Number of free estimable parameters under the proposed model}$. If $Y_1, Y_2 = 1, 2$ and boundary solutions occur, then the boundary MLE's are obtained for the level of Y_1 or Y_2 corresponding to which G^2 is minimum.

Remark 2.1. When $Y_3 = k$ is fixed, consider the $Y_1 Y_2$ -marginal odds ratios. Let $OR_{..k} = (\hat{\pi}_{ijk..} \hat{\pi}_{i'j'k..}) / (\hat{\pi}_{ij'k..} \hat{\pi}_{i'jk..})$ denote an estimated odds ratio on the $Y_1 Y_2$ -margin, where $1 \leq i < i' \leq I$, $1 \leq j < j' \leq J$ and $1 \leq k \leq K$. Also, let $OR_{11k} = (y_{ijk11} y_{i'j'k11}) / (y_{ij'k11} y_{i'jk11})$ be the estimated odds ratio when $R_1 = R_2 = 1$. From the closed-form MLE's in the above models, it can be shown that $OR_{..k} = OR_{11k}$ under models 2, 4, 9, 13 and 16 *a priori*, and under models 3, 5, 6 and 11 for non-boundary (interior) estimates.

Variance estimates. We can derive closed-form estimates for the asymptotic variance in case of non-boundary MLE's. We assume that the data follows Poisson distribution. The asymptotic variance of a statistic $f(\{y_{ijk11}\}, \{y_{i+k12}\}, \{y_{+jk21}\}, y_{++k22})$ for fixed k (see Baker (1994)) is

$$\begin{aligned}
Var(f) &= \sum_{i,j} \left(\frac{\partial f}{\partial y_{ijk11}} \right)^2 \hat{\mu}_{ijk11} + \sum_i \left(\frac{\partial f}{\partial y_{i+k12}} \right)^2 \hat{\mu}_{i+k12} + \sum_j \left(\frac{\partial f}{\partial y_{+jk21}} \right)^2 \hat{\mu}_{+jk21} \\
&\quad + \left(\frac{\partial f}{\partial y_{++k22}} \right)^2 \hat{\mu}_{++k22}.
\end{aligned}
\tag{7}$$

When $OR_{..k} = OR_{11k} = (y_{ijk11} y_{i'j'k11}) / (y_{ij'k11} y_{i'jk11})$, from (7), we get

$$Var(OR_{..k}) = OR_{..k}^2 \left[\frac{\hat{m}_{ijk11}}{y_{ijk11}^2} + \frac{\hat{m}_{ij'k11}}{y_{ij'k11}^2} + \frac{\hat{m}_{i'jk11}}{y_{i'jk11}^2} + \frac{\hat{m}_{i'j'k11}}{y_{i'j'k11}^2} \right].
\tag{8}$$

Using (8), the asymptotic variances of estimated odds ratios for k fixed under various models are as follows.

1. Models 2, 3 and 4 :

$$Var(OR_{..k}) = OR_{..k}^2 \frac{y_{++k11}}{y_{++k+1}} \left[\frac{y_{+jk+1}}{y_{+jk11}} \left(\frac{1}{y_{ijk11}} + \frac{1}{y_{i'jk11}} \right) + \frac{y_{+j'k+1}}{y_{+j'k11}} \left(\frac{1}{y_{ij'k11}} + \frac{1}{y_{i'j'k11}} \right) \right]$$

2. Models 5, 9 and 13 :

$$Var(OR_{..k}) = OR_{..k}^2 \frac{y_{++k11}}{y_{++k1+}} \left[\frac{y_{i+k1+}}{y_{i+k11}} \left(\frac{1}{y_{ijk11}} + \frac{1}{y_{ij'k11}} \right) + \frac{y_{i'+k1+}}{y_{i'+k11}} \left(\frac{1}{y_{i'jk11}} + \frac{1}{y_{i'j'k11}} \right) \right]$$

3. Models 6, 11 and 16 :

$$Var(OR_{..k}) = OR_{..k}^2 \left[\frac{1}{y_{ijk11}} + \frac{1}{y_{ij'k11}} + \frac{1}{y_{i'jk11}} + \frac{1}{y_{i'j'k11}} \right]$$

Alternatively, the variances can be computed from the inverse of the observed information matrix using the method in Baker (1991). Note that in case of boundary MLE's, this method provides a conditional variance estimate, while the bootstrap technique provides an unconditional variance estimate.

2.3. All three variables are missing. For $i = 1, 2, 3$, let R_i denote the missing indicator of Y_i such that $R_i = 1$ if Y_i is observed and $R_i = 2$ otherwise. Then for Y_1, Y_2, Y_3, R_1, R_2 and R_3 , we have an $I \times J \times K \times 2 \times 2 \times 2$ table with cell counts $\mathbf{y} = \{y_{ijkxs}\}$, where $1 \leq i \leq I, 1 \leq j \leq J, 1 \leq k \leq K$ and $x, y, z = 1, 2$. Also, $\mathbf{y}_{\text{obs}} = (\{y_{ijk111}\}, \{y_{+jk211}\}, \{y_{i+k121}\}, \{y_{ij+112}\}, \{y_{++k221}\}, \{y_{+j+212}\}, \{y_{i++122}\}, y_{+++222})$. Let $\pi = \{\pi_{ijkxs}\}$ be the vector of cell probabilities and N be the total cell count. Then the vector of expected counts is $\mu = \{\mu_{ijkxs}\}$, where $\mu_{ijkxs} = N\pi_{ijkxs}$. For $I = J = K = 2$, the $2 \times 2 \times 2 \times 2 \times 2 \times 2$ incomplete table is given below.

Table 3. $2 \times 2 \times 2 \times 2 \times 2 \times 2$ Incomplete Table.

				$R_3 = 1$ $Y_3 = 1$	$Y_3 = 2$	$R_3 = 2$ Missing
$R_1 = 1$	$Y_1 = 1$	$R_2 = 1$	$Y_2 = 1$	y_{111111}	y_{112111}	y_{11+112}
			$Y_2 = 2$	y_{121111}	y_{122111}	y_{12+112}
		$R_2 = 2$	Missing	y_{1+1121}	y_{1+2121}	y_{1++122}
	$Y_1 = 2$	$R_2 = 1$	$Y_2 = 1$	y_{211111}	y_{212111}	y_{21+112}
			$Y_2 = 2$	y_{212111}	y_{222111}	y_{22+112}
		$R_2 = 2$	Missing	y_{2+1121}	y_{2+2121}	y_{2++122}
$R_1 = 2$	Missing	$R_2 = 1$	$Y_2 = 1$	y_{+11211}	y_{+12211}	y_{+1+212}
			$Y_2 = 2$	y_{+21211}	y_{+22211}	y_{+2+212}
		$R_2 = 2$	Missing	y_{++1221}	y_{++2221}	y_{+++222}

The log-linear model (with no three-way or higher order interactions) in this case is

$$\begin{aligned} \log \mu_{ijkxs} = & \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_3}(k) + \lambda_{R_1}(x) + \lambda_{R_2}(s) + \lambda_{R_3}(z) + \lambda_{Y_1 Y_2}(i, j) \\ & + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_2 Y_3}(j, k) + \lambda_{Y_1 R_1}(i, x) + \lambda_{Y_2 R_1}(j, x) + \lambda_{Y_3 R_1}(k, x) + \lambda_{Y_1 R_2}(i, s) \\ & + \lambda_{Y_2 R_2}(j, s) + \lambda_{Y_3 R_2}(k, s) + \lambda_{Y_1 R_3}(i, z) + \lambda_{Y_2 R_3}(j, z) + \lambda_{Y_3 R_3}(k, z) + \lambda_{R_1 R_2}(x, s) \\ (9) \quad & + \lambda_{R_1 R_3}(x, z) + \lambda_{R_2 R_3}(s, z). \end{aligned}$$

Each log-linear parameter in (9) satisfies the constraint that the sum over each of its arguments is 0. Define the following quantities

$$\begin{aligned} a_{ijk} &= \frac{P(R_1 = 2, R_2 = 1, R_3 = 1 \mid Y_1 = i, Y_2 = j, Y_3 = k)}{P(R_1 = 1, R_2 = 1, R_3 = 1 \mid Y_1 = i, Y_2 = j, Y_3 = k)} = \frac{\pi_{ijk211}}{\pi_{ijk111}} = \frac{\mu_{ijk211}}{\mu_{ijk111}}, \\ b_{ijk} &= \frac{P(R_1 = 1, R_2 = 2, R_3 = 1 \mid Y_1 = i, Y_2 = j, Y_3 = k)}{P(R_1 = 1, R_2 = 1, R_3 = 1 \mid Y_1 = i, Y_2 = j, Y_3 = k)} = \frac{\pi_{ijk121}}{\pi_{ijk111}} = \frac{\mu_{ijk121}}{\mu_{ijk111}}, \\ c_{ijk} &= \frac{P(R_1 = 1, R_2 = 1, R_3 = 2 \mid Y_1 = i, Y_2 = j, Y_3 = k)}{P(R_1 = 1, R_2 = 1, R_3 = 1 \mid Y_1 = i, Y_2 = j, Y_3 = k)} = \frac{\pi_{ijk112}}{\pi_{ijk111}} = \frac{\mu_{ijk112}}{\mu_{ijk111}}. \end{aligned}$$

Then a_{ijk} , b_{ijk} and c_{ijk} describe the missing data mechanisms of Y_1 , Y_2 and Y_3 respectively. Here a_{ijk} is the conditional odds of Y_1 being missing given both Y_2 and Y_3 are observed, b_{ijk} is the conditional odds of Y_2 being missing given both Y_1 and Y_3 are observed, and c_{ijk} is the conditional odds of Y_3 being missing given both Y_1 and Y_2 are observed. Let the conditional odds ratio between R_1 and R_2 given Y_3 is observed be

$$\begin{aligned}\theta_{12} &= \frac{P(R_1 = 1, R_2 = 1, R_3 = 1 | Y_1 = i, Y_2 = j, Y_3 = k)}{P(R_1 = 1, R_2 = 2, R_3 = 1 | Y_1 = i, Y_2 = j, Y_3 = k)} \\ &\quad \times \frac{P(R_1 = 2, R_2 = 2, R_3 = 1 | Y_1 = i, Y_2 = j, Y_3 = k)}{P(R_1 = 2, R_2 = 1, R_3 = 1 | Y_1 = i, Y_2 = j, Y_3 = k)} \\ &= \frac{\pi_{ijk111}\pi_{ijk221}}{\pi_{ijk121}\pi_{ijk211}} = \frac{\mu_{ijk111}\mu_{ijk221}}{\mu_{ijk121}\mu_{ijk211}}.\end{aligned}$$

Similarly, define θ_{13} to be the conditional odds ratio between R_1 and R_3 given Y_2 is observed, and θ_{23} to be the conditional odds ratio between R_2 and R_3 given Y_1 is observed. Also, define

$$\begin{aligned}\theta_{123} &= \frac{P(R_1 = 2, R_2 = 2, R_3 = 2 | Y_1 = i, Y_2 = j, Y_3 = k)}{P(R_1 = 1, R_2 = 1, R_3 = 1 | Y_1 = i, Y_2 = j, Y_3 = k)} \\ &= \frac{\pi_{ijk222}}{\pi_{ijk111}} = \frac{\mu_{ijk222}}{\mu_{ijk111}}.\end{aligned}$$

Here, θ_{12} , θ_{13} and θ_{23} describe the conditional associations between the missing mechanisms of Y_1 and Y_2 , Y_1 and Y_3 , and Y_2 and Y_3 respectively. For $i \neq j \neq k = 1, 2, 3$, if $\theta_{ij} = 1$, then the missing mechanisms of Y_i and Y_j are conditionally independent given that Y_k is observed. Note that θ_{123} denotes the joint odds of Y_1, Y_2 and Y_3 simultaneously missing. Let $m_{ijk111} = \mu_{ijk111}$. Then $\mu_{ijk211} = a_{ijk}m_{ijk111}$, $\mu_{ijk121} = b_{ijk}m_{ijk111}$, $\mu_{ijk112} = c_{ijk}m_{ijk111}$, $\mu_{ijk221} = m_{ijk111}a_{ijk}b_{ijk}\theta_{12}$, $\mu_{ijk212} = m_{ijk111}a_{ijk}c_{ijk}\theta_{13}$, $\mu_{ijk122} = m_{ijk111}b_{ijk}c_{ijk}\theta_{23}$, $\mu_{ijk222} = m_{ijk111}\theta_{123}$ and $N = \sum_{i,j,k} m_{ijk111}(1 + a_{ijk} + b_{ijk} + c_{ijk} + a_{ijk}b_{ijk}\theta_{12} + a_{ijk}c_{ijk}\theta_{13} + b_{ijk}c_{ijk}\theta_{23} + \theta_{123})$. The joint probability is $\pi_{ijk..} = m_{ijk111}(1 + a_{ijk} + b_{ijk} + c_{ijk} + a_{ijk}b_{ijk}\theta_{12} + a_{ijk}c_{ijk}\theta_{13} + b_{ijk}c_{ijk}\theta_{23} + \theta_{123})/N$, from which the marginals can be obtained. Under (9), we have

$$\begin{aligned}a_{ijk} &= \exp[-2\{\lambda_{R_1}(1) + \lambda_{Y_1R_1}(i, 1) + \lambda_{Y_2R_1}(j, 1) + \lambda_{Y_3R_1}(k, 1) + \lambda_{R_1R_2}(1, 1) + \lambda_{R_1R_3}(1, 1)\}], \\ b_{ijk} &= \exp[-2\{\lambda_{R_2}(1) + \lambda_{Y_1R_2}(i, 1) + \lambda_{Y_2R_2}(j, 1) + \lambda_{Y_3R_2}(k, 1) + \lambda_{R_1R_2}(1, 1) + \lambda_{R_2R_3}(1, 1)\}], \\ c_{ijk} &= \exp[-2\{\lambda_{R_3}(1) + \lambda_{Y_1R_3}(i, 1) + \lambda_{Y_2R_3}(j, 1) + \lambda_{Y_3R_3}(k, 1) + \lambda_{R_1R_3}(1, 1) + \lambda_{R_2R_3}(1, 1)\}], \\ \theta_{12} &= \exp[4\lambda_{R_1R_2}(1, 1)], \theta_{13} = \exp[4\lambda_{R_1R_3}(1, 1)], \theta_{23} = \exp[4\lambda_{R_2R_3}(1, 1)], \\ \theta_{123} &= \exp[-2\{\lambda_{R_1}(1) + \lambda_{R_2}(1) + \lambda_{R_3}(1) + \lambda_{Y_1R_1}(i, 1) + \lambda_{Y_2R_1}(j, 1) + \lambda_{Y_3R_1}(k, 1) + \lambda_{Y_1R_2}(i, 1) \\ &\quad + \lambda_{Y_2R_2}(j, 1) + \lambda_{Y_3R_2}(k, 1) + \lambda_{Y_1R_3}(i, 1) + \lambda_{Y_2R_3}(j, 1) + \lambda_{Y_3R_3}(k, 1)\}].\end{aligned}$$

Based on the assumption in the previous case regarding the missing mechanism of a variable, each of a_{ijk} , b_{ijk} and c_{ijk} may depend on one of i, j, k or none. The different values assumed by them are $a_{ijk} = (\alpha_{...}, \alpha_{i..}, \alpha_{.j.}, \alpha_{..k})$, $b_{ijk} = (\beta_{...}, \beta_{i..}, \beta_{.j.}, \beta_{..k})$ and $c_{ijk} = (\gamma_{...}, \gamma_{i..}, \gamma_{.j.}, \gamma_{..k})$ (say).

Definition 2.3. The missing mechanism of Y_1 under (9) is NMAR if $a_{ijk} = \alpha_{i..}$, MAR if $a_{ijk} = \alpha_{.j.}$ or $\alpha_{..k}$ and MCAR if $a_{ijk} = \alpha_{...}$. Similarly, the missing mechanism of Y_2 is NMAR if $b_{ijk} = \beta_{.j.}$, MAR if $b_{ijk} = \beta_{i..}$ or $\beta_{..k}$ and MCAR if $b_{ijk} = \beta_{...}$. Finally, the missing mechanism of Y_3 is NMAR if $c_{ijk} = \gamma_{..k}$, MAR if $c_{ijk} = \gamma_{i..}$ or $\gamma_{.j.}$ and MCAR if $c_{ijk} = \gamma_{...}$.

We have 64 possible identifiable models as follows.

1. MCAR model - $(\alpha_{...}, \beta_{...}, \gamma_{...})$,
2. NMAR model - $(\alpha_{i..}, \beta_{.j.}, \gamma_{..k})$,
3. MAR models - $(\alpha_{.j.}, \beta_{i..}, \gamma_{i..}), (\alpha_{.j.}, \beta_{i..}, \gamma_{..j}), (\alpha_{.j.}, \beta_{..k}, \gamma_{i..}), (\alpha_{.j.}, \beta_{..k}, \gamma_{.j.}), (\alpha_{..k}, \beta_{i..}, \gamma_{i..}), (\alpha_{..k}, \beta_{i..}, \gamma_{.j.}), (\alpha_{..k}, \beta_{..k}, \gamma_{i..}), (\alpha_{..k}, \beta_{..k}, \gamma_{.j.})$
4. Mixture of MCAR and NMAR models - $(\alpha_{i..}, \beta_{...}, \gamma_{...}), (\alpha_{...}, \beta_{.j.}, \gamma_{...}), (\alpha_{...}, \beta_{...}, \gamma_{..k}), (\alpha_{i..}, \beta_{.j.}, \gamma_{...}), (\alpha_{i..}, \beta_{...}, \gamma_{..k}), (\alpha_{...}, \beta_{.j.}, \gamma_{..k}), (\alpha_{...}, \beta_{i..}, \gamma_{i..}), (\alpha_{...}, \beta_{i..}, \gamma_{.j.}), (\alpha_{...}, \beta_{..k}, \gamma_{i..}), (\alpha_{...}, \beta_{..k}, \gamma_{.j.}), (\alpha_{.j.}, \beta_{...}, \gamma_{i..}), (\alpha_{.j.}, \beta_{...}, \gamma_{.j.}), (\alpha_{..k}, \beta_{...}, \gamma_{i..}), (\alpha_{..k}, \beta_{...}, \gamma_{.j.}), (\alpha_{.j.}, \beta_{i..}, \gamma_{...}), (\alpha_{.j.}, \beta_{..k}, \gamma_{...}), (\alpha_{..k}, \beta_{i..}, \gamma_{...}), (\alpha_{..k}, \beta_{..k}, \gamma_{...}), (\alpha_{...}, \beta_{...}, \gamma_{i..}), (\alpha_{...}, \beta_{...}, \gamma_{.j.}), (\alpha_{...}, \beta_{i..}, \gamma_{...}), (\alpha_{...}, \beta_{..k}, \gamma_{...}), (\alpha_{.j.}, \beta_{...}, \gamma_{...}), (\alpha_{..k}, \beta_{...}, \gamma_{...})$,
5. Mixture of MCAR and MAR models - $(\alpha_{...}, \beta_{i..}, \gamma_{i..}), (\alpha_{...}, \beta_{i..}, \gamma_{.j.}), (\alpha_{...}, \beta_{..k}, \gamma_{i..}), (\alpha_{...}, \beta_{..k}, \gamma_{.j.}), (\alpha_{.j.}, \beta_{...}, \gamma_{i..}), (\alpha_{.j.}, \beta_{...}, \gamma_{.j.}), (\alpha_{..k}, \beta_{...}, \gamma_{i..}), (\alpha_{..k}, \beta_{...}, \gamma_{.j.}), (\alpha_{.j.}, \beta_{i..}, \gamma_{...}), (\alpha_{.j.}, \beta_{..k}, \gamma_{...}), (\alpha_{..k}, \beta_{i..}, \gamma_{...}), (\alpha_{..k}, \beta_{..k}, \gamma_{...}), (\alpha_{...}, \beta_{...}, \gamma_{i..}), (\alpha_{...}, \beta_{...}, \gamma_{.j.}), (\alpha_{...}, \beta_{i..}, \gamma_{...}), (\alpha_{...}, \beta_{..k}, \gamma_{...}), (\alpha_{.j.}, \beta_{...}, \gamma_{...}), (\alpha_{..k}, \beta_{...}, \gamma_{...})$,
6. Mixture of NMAR and MAR models - $(\alpha_{i..}, \beta_{i..}, \gamma_{i..}), (\alpha_{i..}, \beta_{i..}, \gamma_{.j.}), (\alpha_{i..}, \beta_{..k}, \gamma_{i..}), (\alpha_{i..}, \beta_{..k}, \gamma_{.j.}), (\alpha_{.j.}, \beta_{.j.}, \gamma_{i..}), (\alpha_{.j.}, \beta_{.j.}, \gamma_{.j.}), (\alpha_{..k}, \beta_{.j.}, \gamma_{i..}), (\alpha_{..k}, \beta_{.j.}, \gamma_{.j.}), (\alpha_{.j.}, \beta_{i..}, \gamma_{..k}), (\alpha_{.j.}, \beta_{..k}, \gamma_{..k}), (\alpha_{..k}, \beta_{i..}, \gamma_{..k}), (\alpha_{..k}, \beta_{..k}, \gamma_{..k}), (\alpha_{i..}, \beta_{.j.}, \gamma_{i..}), (\alpha_{i..}, \beta_{.j.}, \gamma_{.j.}), (\alpha_{i..}, \beta_{i..}, \gamma_{..k}), (\alpha_{i..}, \beta_{..k}, \gamma_{..k}), (\alpha_{.j.}, \beta_{.j.}, \gamma_{..k}), (\alpha_{..k}, \beta_{.j.}, \gamma_{..k})$,
7. Mixture of NMAR, MAR and MCAR models - $(\alpha_{i..}, \beta_{i..}, \gamma_{...}), (\alpha_{i..}, \beta_{..k}, \gamma_{...}), (\alpha_{i..}, \beta_{...}, \gamma_{i..}), (\alpha_{i..}, \beta_{...}, \gamma_{.j.}), (\alpha_{.j.}, \beta_{.j.}, \gamma_{...}), (\alpha_{..k}, \beta_{.j.}, \gamma_{...}), (\alpha_{...}, \beta_{.j.}, \gamma_{i..}), (\alpha_{...}, \beta_{.j.}, \gamma_{.j.}), (\alpha_{.j.}, \beta_{...}, \gamma_{..k}), (\alpha_{..k}, \beta_{...}, \gamma_{..k}), (\alpha_{...}, \beta_{i..}, \gamma_{..k}), (\alpha_{...}, \beta_{..k}, \gamma_{..k})$.

The log-likelihood kernel of μ under Poisson sampling is

$$\begin{aligned}
 l(\mu; \mathbf{y}_{\text{obs}}) = & \sum_{i,j,k} y_{ijk111} \log \mu_{ijk111} + \sum_{j,k} y_{+jk211} \log \mu_{+jk211} + \sum_{i,k} y_{i+k121} \log \mu_{i+k121} \\
 & + \sum_{i,j} y_{ij+112} \log \mu_{ij+112} + \sum_k y_{++k221} \log \mu_{++k221} \\
 & + \sum_j y_{+j+212} \log \mu_{+j+212} + \sum_i y_{i++122} \log \mu_{i++122} \\
 (10) \quad & + y_{+++222} \log \mu_{+++222} - \sum_{i,j,k,x,s,z} \mu_{ijkxs z}.
 \end{aligned}$$

Rewriting (10) in terms of $a_{ijk}, b_{ijk}, c_{ijk}, \theta_{12}, \theta_{13}, \theta_{23}$ and θ_{123} , we obtain the log-likelihood as a function of the unknown parameters, after which ML estimation can be performed. We obtain closed-form MLE's of m_{ijk111} under various missing data models given above (model categories 1 to 7) except for the model $(\alpha_{...}, \beta_{...}, \gamma_{...})$, in which case the EM algorithm is used. Note that boundary solutions occur if at least one of the following holds.

1. $\hat{\alpha}_{i..} \leq 0$ for at least one and at most $(I - 1)$ values of Y_1 ,
2. $\hat{\beta}_{.j.} \leq 0$ for at least one and at most $(J - 1)$ values of Y_2 ,
3. $\hat{\gamma}_{..k} \leq 0$ for at least one and at most $(K - 1)$ values of Y_3 .

If any $\hat{\alpha}_{i..} < 0$, $\hat{\beta}_{.j.} < 0$ or $\hat{\gamma}_{..k} < 0$, then boundary estimates are obtained by setting $\hat{\alpha}_{i..} = \hat{\beta}_{.j.} = \hat{\gamma}_{..k} = 0$ in (10) for relevant models.

Consider the following hypotheses : H_0 - the proposed model (among model categories 1 to 7 mentioned above) fits the data, and H_1 - the perfect fit model fits the data. Let L_0 and L_1 denote the maximized log-likelihood functions under the proposed and perfect fit models

respectively. Then the likelihood ratio statistic for testing H_0 against H_1 is

$$\begin{aligned}
G^2 &= -2(L_0 - L_1) \\
&= -2 \left[\sum_{i,j,k} y_{ijk111} \ln \left(\frac{\hat{m}_{ijk111}}{y_{ijk111}} \right) + \sum_{j,k} y_{+jk211} \ln \left(\frac{\sum_i \hat{m}_{ijk111} \hat{a}_{ijk}}{y_{+jk211}} \right) \right. \\
&\quad + \sum_{i,k} y_{i+k121} \ln \left(\frac{\sum_j \hat{m}_{ijk111} \hat{b}_{ijk}}{y_{i+k121}} \right) + \sum_{i,j} y_{ij+112} \ln \left(\frac{\sum_k \hat{m}_{ijk111} \hat{c}_{ijk}}{y_{ij+112}} \right) \\
&\quad + \sum_k y_{++k221} \ln \left(\frac{\sum_{i,j} \hat{m}_{ijk111} \hat{a}_{ijk} \hat{b}_{ijk} \theta_{12}}{y_{++k221}} \right) + \sum_j y_{+j+212} \ln \left(\frac{\sum_{i,k} \hat{m}_{ijk111} \hat{a}_{ijk} \hat{c}_{ijk} \theta_{13}}{y_{+j+212}} \right) \\
&\quad \left. + \sum_i y_{i++122} \ln \left(\frac{\sum_{j,k} \hat{m}_{ijk111} \hat{b}_{ijk} \hat{c}_{ijk} \theta_{23}}{y_{i++122}} \right) + N \right] \\
(11) \quad &- 2 \sum_{i,j,k} \hat{m}_{ijk111} (1 + \hat{a}_{ijk} + \hat{b}_{ijk} + \hat{c}_{ijk} + \hat{a}_{ijk} \hat{b}_{ijk} \hat{\theta}_{12} + \hat{a}_{ijk} \hat{c}_{ijk} \hat{\theta}_{13} + \hat{b}_{ijk} \hat{c}_{ijk} \hat{\theta}_{23} + \hat{\theta}_{123}).
\end{aligned}$$

Here G^2 follows χ^2_ν asymptotically, where $\nu = (I+1)(J+1)(K+1) - \text{Number of free estimable parameters under the proposed model}$.

Remark 2.2. From all the above cases, note that perfect fit solutions for fully observed counts occur for the following types of models:

1. non-boundary cases of NMAR only models for one or more variables,
2. non-boundary cases of a mixture of NMAR and MAR models for the variables,
3. MAR only models for two or more variables.

However, if the missing mechanism is MCAR for at least one of the variables, then perfect fit solutions don't occur.

Without loss of generality, consider NMAR models for Y_1 in $I \times J \times K \times 2$, $I \times J \times K \times 2 \times 2$ and $I \times J \times K \times 2 \times 2 \times 2$ tables. Then we have the following remarks.

Remark 2.3. The systems of equations $\sum_i \hat{m}_{ijk1} \hat{\alpha}_{i..} = y_{+jk2}$, $\sum_i \hat{m}_{ijk11} \hat{\alpha}_{i..} = y_{+jk21}$ and $\sum_i \hat{m}_{ijk111} \hat{\alpha}_{i...} = y_{+jk211}$ in $I \times J \times K \times 2$, $I \times J \times K \times 2 \times 2$ and $I \times J \times K \times 2 \times 2 \times 2$ tables respectively are overdetermined (underdetermined) if $I < JK$ ($I > JK$).

Remark 2.4. Let $A = (\hat{m}_{ijk1})$ or $A = (\hat{m}_{ijk11})$ or $A = (\hat{m}_{ijk111})$ for $I \times J \times K \times 2$ or $I \times J \times K \times 2 \times 2$ or $I \times J \times K \times 2 \times 2 \times 2$ tables respectively.

1. From Remark 2.3, A is square if $I = JK$ and rectangular otherwise. If A is square and non-singular, then unique MLE's of $\alpha_{i..}$ exist.
2. Consider overdetermined systems in Remark 2.3. Unique solutions (MLE's of $\alpha_{i..}$) exist provided A is of full rank, that is, $\text{rank}(A) = I$. Then the left inverse of A given by $A_{\text{left}}^{-1} = (A^T A)^{-1} A^T$ exists. The solutions are obtained using the method of ordinary least squares (see Williams (1990)).
3. Consider underdetermined systems in Remark 2.3. Unique solutions (MLE's of $\alpha_{i..}$) if A is of full rank, that is, $\text{rank}(A) = JK$. Then the right inverse of A given by $A_{\text{right}}^{-1} = A^T (A A^T)^{-1}$ exists. The solutions are obtained using the method of minimum norm least squares (see Madych (1991)).

3. IDENTIFIABLE NMAR LOG-LINEAR MODELS

In this section, we consider several identifiable NMAR models for $I \times J \times K \times 2$, $I \times J \times K \times 2 \times 2$ and $I \times J \times K \times 2 \times 2 \times 2$ incomplete tables. These models deal with nonignorable nonresponse where the missing data mechanism of some variable depends on the variable itself. We assume Y_1 is missing in an $I \times J \times K \times 2$ table, and Y_1 , Y_2 are missing in an $I \times J \times K \times 2 \times 2$ table. From the previous section, the various identifiable NMAR models for each of the three cases are listed below.

- (a) $I \times J \times K \times 2$ table - model 1 (see Subsection 2.1),
- (b) $I \times J \times K \times 2 \times 2$ table - models 3, 5, 6, 7, 8, 11 and 15 (see Subsection 2.2),
- (c) $I \times J \times K \times 2 \times 2 \times 2$ table - models belonging to categories 2, 4, 6 and 7 (see Subsection 2.3).

4. OCCURRENCE OF BOUNDARY SOLUTIONS IN THE NMAR MODELS

In this section, we first describe boundary solutions under NMAR models in three-way incomplete contingency tables with one, two or all three variables missing. A sufficient condition for the occurrence of boundary solutions for each of the above types of tables is then established. This condition depends only on the observed counts and not the ML estimates obtained using the EM algorithm. We generalize the results given by Park *et al.* (2014) to the above cases.

4.1. Boundary solutions in the NMAR models. Boundary solutions under NMAR models in an incomplete table occur when the ML estimates of nonresponse cell probabilities are all zeros for some levels of a variable, that is, they lie on the boundary of the parameter space. For an $I \times J \times 2$ table with Y_2 missing, Baker and Laird (1988) defined boundary solutions as $\hat{\pi}_{ij2} = 0$ for at least one pair (i, j) . Clarke and Smith (2005) suggested their form as $\hat{\pi}_{+j2} = 0$ for at least one and at most $(J - 1)$ values of Y_2 when Y_2 is missing. For an $I \times J \times 2 \times 2$ table, Park *et al.* (2014) defined boundary solutions as having at least one of the following forms : (i) $\hat{\pi}_{i+2+} = 0$ for at least one and at most $(I - 1)$ values of Y_1 and (ii) $\hat{\pi}_{+j+2} = 0$ for at least one and at most $(J - 1)$ values of Y_2 . We now consider the boundary solutions for different three-way incomplete tables as summarized below. Without loss of generality, it is assumed Y_1 is missing in an $I \times J \times K \times 2$ table, and Y_1 and Y_2 are missing in an $I \times J \times K \times 2 \times 2$ table.

(i) $I \times J \times K \times 2$ table.

Y_1 missing - The boundary solutions in the NMAR model for Y_1 are of the form $\hat{\pi}_{i++2} = 0$ for at least one and at most $(I - 1)$ values of Y_1 .

(ii) $I \times J \times K \times 2 \times 2$ table.

Y_1 and Y_2 missing - The boundary solutions in the NMAR model for both Y_1 and Y_2 take at least one of the following forms :

1. $\hat{\pi}_{i++2+} = 0$ for at least one and at most $(I - 1)$ values of Y_1 ,
2. $\hat{\pi}_{++j+2} = 0$ for at least one and at most $(J - 1)$ values of Y_2 .

(iii) $I \times J \times K \times 2 \times 2 \times 2$ table.

Y_1 , Y_2 and Y_3 missing - The boundary solutions in the NMAR model for Y_1 , Y_2 and Y_3 take at least one of the following forms :

1. $\hat{\pi}_{i+++2+} = 0$ for at least one and at most $(I - 1)$ values of Y_1 ,

2. $\hat{\pi}_{++2+} = 0$ for at least one and at most $(J - 1)$ values of Y_2 ,
3. $\hat{\pi}_{++k+2} = 0$ for at least one and at most $(K - 1)$ values of Y_3 .

4.2. Conditions for the occurrence of boundary solutions. In this section, we propose sufficient conditions for boundary solutions in NMAR models listed above for $I \times J \times K \times 2$, $I \times J \times K \times 2 \times 2$ and $I \times J \times K \times 2 \times 2 \times 2$ tables.

(i) $I \times J \times K \times 2$ table.

Case 1: Y_1 missing. Define the following odds for any pair (j, j') of Y_2 :

(12)

$$\nu_{ik}(j, j') = \frac{\hat{\pi}_{ijk1}}{\hat{\pi}_{ij'k1}}, \quad \nu_{nk}(j, j') = \min_i \{\nu_{ik}(j, j')\}, \quad \nu_{nk}(j, j') = \max_i \{\nu_{ik}(j, j')\}, \quad \nu_k(j, j') = \frac{y_{+jk2}}{y_{+j'k2}}.$$

Note that the fully observed counts may be termed response odds, while the partially classified margins may be termed nonresponse odds. Similarly, for any pair (k, k') of Y_3 , define the following odds :

(13)

$$\nu_{ij}(k, k') = \frac{\hat{\pi}_{ijk1}}{\hat{\pi}_{ij'k1}}, \quad \nu_{nj}(k, k') = \min_i \{\nu_{ij}(k, k')\}, \quad \nu_{mj}(k, k') = \max_i \{\nu_{ij}(k, k')\}, \quad \nu_j(k, k') = \frac{y_{+jk2}}{y_{+j'k2}}.$$

The following theorem gives a sufficient condition for the occurrence of boundary solutions in this case.

Theorem 4.1. For an $I \times J \times K \times 2$ incomplete table, boundary solutions in the NMAR model for Y_1 occur if both the following conditions hold:

1. $\nu_k(j, j') \notin (\nu_{nk}(j, j'), \nu_{mk}(j, j'))$ for at least one pair (j, j') of Y_2 ,
2. $\nu_j(k, k') \notin (\nu_{nj}(k, k'), \nu_{mj}(k, k'))$ for at least one pair (k, k') of Y_3 .

Proof. The MLE $\hat{\alpha}_{i..}$ under the NMAR model for Y_1 satisfies

$$(14) \quad \sum_i N \hat{\pi}_{ijk1} \hat{\alpha}_{i..} = y_{+jk2} \quad \forall 1 \leq j \leq J, 1 \leq k \leq K,$$

where $\hat{\pi}_{ijk1} = y_{ijk1}/N$ is the MLE of π_{ijk1} in case of interior (non-boundary) solutions. From (12), (13) and (14), we have

$$(15) \quad \nu_k(j, j') = \frac{y_{+jk2}}{y_{+j'k2}} = \frac{\sum_i \hat{\pi}_{ijk1} \hat{\alpha}_{i..}}{\sum_i \hat{\pi}_{ij'k1} \hat{\alpha}_{i..}},$$

$$\nu_{mk}(j, j') - \nu_k(j, j') = \frac{\sum_{i \neq m_1} (\hat{\pi}_{m_1 j k 1} \hat{\pi}_{ij' k 1} - \hat{\pi}_{m_1 j' k 1} \hat{\pi}_{ij k 1}) \hat{\alpha}_{i..}}{\hat{\pi}_{m_1 j' k 1} \sum_i \hat{\pi}_{ij' k 1} \hat{\alpha}_{i..}},$$

$$(16) \quad \nu_k(j, j') - \nu_{nk}(j, j') = \frac{\sum_{i \neq n_1} (\hat{\pi}_{n_1 j' k 1} \hat{\pi}_{ij k 1} - \hat{\pi}_{n_1 j k 1} \hat{\pi}_{ij' k 1}) \hat{\alpha}_{i..}}{\hat{\pi}_{n_1 j' k 1} \sum_i \hat{\pi}_{ij' k 1} \hat{\alpha}_{i..}},$$

where m_1 and n_1 are the levels of Y_1 corresponding to $\nu_{mk}(j, j')$ and $\nu_{nk}(j, j')$ respectively. From (12),

$$(17) \quad \nu_{nk}(j, j') = \frac{\hat{\pi}_{n_1 j k 1}}{\hat{\pi}_{n_1 j' k 1}} < \nu_{ik}(j, j') = \frac{\hat{\pi}_{ijk1}}{\hat{\pi}_{ij'k1}} < \nu_{mk}(j, j') = \frac{\hat{\pi}_{m_1 j k 1}}{\hat{\pi}_{m_1 j' k 1}}.$$

From (17), we have the following inequalities

$$(18) \quad \hat{\pi}_{m_1jk1}\hat{\pi}_{ij'k1} > \hat{\pi}_{m_1j'k1}\hat{\pi}_{ijk1}, \quad \hat{\pi}_{n_1j'k1}\hat{\pi}_{ijk1} > \hat{\pi}_{n_1jk1}\hat{\pi}_{ij'k1} \text{ for } i \neq m_1, n_1.$$

Now consider condition 1 which implies that (15) and (16) are of opposite signs. Using this fact and (18), we observe that $\hat{\alpha}_{i..} < 0$ for at least one and at most $(I - 1)$ values of Y_1 , that is, boundary solutions of type $\hat{\pi}_{i++2} = 0$ occur. Using similar arguments, it can be shown that condition 2 also implies the same. This completes the proof. \square

Case 2: Y_2 missing. Define the following odds for any pair (i, i') of Y_1 :

$$(19) \quad \omega_{jk}(i, i') = \frac{\hat{\pi}_{ijk1}}{\hat{\pi}_{i'jk1}}, \quad \omega_{nk}(i, i') = \min_j \{\omega_{jk}(i, i')\}, \quad \omega_{mk}(i, i') = \max_j \{\omega_{jk}(i, i')\}, \quad \omega_k(i, i') = \frac{y_{i+k2}}{y_{i'+k2}}.$$

Similarly, for any pair (k, k') of Y_3 , define the following odds :

$$(20) \quad \omega_{ji}(k, k') = \frac{\hat{\pi}_{ijk1}}{\hat{\pi}_{ijk'1}}, \quad \omega_{ni}(k, k') = \min_j \{\omega_{ji}(k, k')\}, \quad \omega_{mi}(k, k') = \max_j \{\omega_{ji}(k, k')\}, \quad \omega_i(k, k') = \frac{y_{i+k2}}{y_{i+k'2}}.$$

A sufficient condition for the occurrence of boundary solutions in this case can be obtained as in Theorem 4.1 using the odds in (19) and (20).

Case 3: Y_3 missing. Define the following odds for any pair (i, i') of Y_1 :

$$(21) \quad \delta_{kj}(i, i') = \frac{\hat{\pi}_{ijk1}}{\hat{\pi}_{i'jk1}}, \quad \delta_{nj}(i, i') = \min_k \{\delta_{kj}(i, i')\}, \quad \delta_{mj}(i, i') = \max_k \{\delta_{kj}(i, i')\}, \quad \delta_j(i, i') = \frac{y_{ij+2}}{y_{i'j+2}}.$$

Similarly, for any pair (j, j') of Y_2 , define the following odds :

$$(22) \quad \delta_{ki}(j, j') = \frac{\hat{\pi}_{ijk1}}{\hat{\pi}_{ijk'1}}, \quad \delta_{ni}(j, j') = \min_k \{\delta_{ki}(j, j')\}, \quad \delta_{mi}(j, j') = \max_k \{\delta_{ki}(j, j')\}, \quad \delta_i(j, j') = \frac{y_{ij+2}}{y_{ij'+2}}.$$

A sufficient condition for the occurrence of boundary solutions in this case can be provided as in Theorem 4.1 using the odds in (21) and (22).

(ii) $I \times J \times K \times 2 \times 2$ table.

Define the odds $\nu'_{ik}(j, j')$, $\nu'_{nk}(j, j')$, $\nu'_{mk}(j, j')$, $\nu'_k(j, j')$, $\nu'_{ij}(k, k')$, $\nu'_{nj}(k, k')$, $\nu'_{mj}(k, k')$ and $\nu'_j(k, k')$ similarly as in (12) and (13). In this case, replace $\hat{\pi}_{ijk1}$ by $\hat{\pi}_{ijk11}$, $\hat{\pi}_{ij'k1}$ by $\hat{\pi}_{ij'k11}$, $\hat{\pi}_{ijk'1}$ by $\hat{\pi}_{ijk'11}$, y_{+jk2} by y_{+jk21} , $y_{+j'k2}$ by $y_{+j'k21}$ and $y_{+jk'2}$ by $y_{+jk'21}$.

Also, define the odds $\omega'_{jk}(i, i')$, $\omega'_{nk}(i, i')$, $\omega'_{mk}(i, i')$, $\omega'_k(i, i')$, $\omega'_{ji}(k, k')$, $\omega'_{ni}(k, k')$, $\omega'_{mi}(k, k')$ and $\omega'_i(k, k')$ similarly as in (19) and (20). In this case, replace $\hat{\pi}_{ijk1}$ by $\hat{\pi}_{ijk11}$, $\hat{\pi}_{ij'k1}$ by $\hat{\pi}_{ij'k11}$, $\hat{\pi}_{ijk'1}$ by $\hat{\pi}_{ijk'11}$, y_{i+k2} by y_{i+k12} , $y_{i'+k2}$ by $y_{i'+k12}$ and $y_{i+k'2}$ by $y_{i+k'12}$. Without loss of generality, assume Y_1 and Y_2 are missing. The theorem below gives sufficient conditions for the occurrence of boundary solutions in this case.

Theorem 4.2. For an $I \times J \times K \times 2 \times 2$ incomplete table, consider the following sets (1a,1b) and (2a,2b) of conditions:

- 1a. $\nu'_k(j, j') \notin (\nu'_{nk}(j, j'), \nu'_{mk}(j, j'))$ for at least one pair (j, j') of Y_2 ,
- 1b. $\nu'_j(k, k') \notin (\nu'_{nj}(k, k'), \nu'_{mj}(k, k'))$ for at least one pair (k, k') of Y_3 .

- 2a. $\omega'_k(i, i') \notin (\omega'_{nk}(i, i'), \omega'_{mk}(i, i'))$ for at least one pair (i, i') of Y_1 ,
 2b. $\omega'_i(k, k') \notin (\omega'_{ni}(k, k'), \omega'_{mi}(k, k'))$ for at least one pair (k, k') of Y_3 .

Then we have the following:

- (a) Boundary solutions in NMAR models for only Y_1 occur if the set (1a,1b) of conditions holds.
- (b) Boundary solutions in NMAR models for only Y_2 occur if the set (2a,2b) of conditions holds.
- (c) Boundary solutions in the NMAR model for both Y_1 and Y_2 occur if set (1a,1b) or set (2a,2b) of conditions holds.

Proof. The MLE's $\hat{\alpha}_{i..}$ and $\hat{\beta}_{.j}$ under the NMAR model for both Y_1 and Y_2 satisfy

$$(23) \quad \sum_i N \hat{\pi}_{ijk11} \hat{\alpha}_{i..} = y_{+jk21} \quad \forall 1 \leq j \leq J, 1 \leq k \leq K,$$

$$(24) \quad \sum_j N \hat{\pi}_{ijk11} \hat{\beta}_{.j} = y_{i+k12} \quad \forall 1 \leq i \leq I, 1 \leq k \leq K.$$

where $\hat{\pi}_{ijk11} = y_{ijk11}/N$ is the MLE of π_{ijk11} in case of interior (non-boundary) solutions. The MLE $\hat{\alpha}_{i..}$ under the NMAR models for only Y_1 satisfies (23), while the MLE $\hat{\beta}_{.j}$ under the NMAR models for only Y_2 satisfies (24). Also note that boundary solutions under the NMAR model for both Y_1 and Y_2 occur if at least one of the following holds:

- (i) $\hat{\alpha}_{i..} \leq 0$ for at least one and at most $(I - 1)$ values of Y_1 ,
- (ii) $\hat{\beta}_{.j} \leq 0$ for at least one and at most $(J - 1)$ values of Y_2 .

Boundary solutions under NMAR models for only Y_1 occur if Condition (i) holds, while boundary solutions under NMAR models for only Y_2 occur if Condition (ii) holds. We have

$$(25) \quad \nu'_k(j, j') = \frac{y_{+jk21}}{y_{+j'k21}} = \frac{\sum_i \hat{\pi}_{ijk11} \hat{\alpha}_{i..}}{\sum_i \hat{\pi}_{ij'k11} \hat{\alpha}_{i..}},$$

$$\nu'_{mk}(j, j') - \nu'_k(j, j') = \frac{\sum_{i \neq m_1} (\hat{\pi}_{m_1jk11} \hat{\pi}_{ij'k11} - \hat{\pi}_{m_1j'k11} \hat{\pi}_{ijk11}) \hat{\alpha}_{i..}}{\hat{\pi}_{m_1j'k11} \sum_i \hat{\pi}_{ij'k11} \hat{\alpha}_{i..}},$$

$$(26) \quad \nu'_k(j, j') - \nu'_{nk}(j, j') = \frac{\sum_{i \neq n_1} (\hat{\pi}_{n_1j'k11} \hat{\pi}_{ijk11} - \hat{\pi}_{n_1jk11} \hat{\pi}_{ij'k11}) \hat{\alpha}_{i..}}{\hat{\pi}_{n_1j'k11} \sum_i \hat{\pi}_{ij'k11} \hat{\alpha}_{i..}},$$

where m_1 and n_1 are the levels of Y_1 corresponding to $\nu'_{mk}(j, j')$ and $\nu'_{nk}(j, j')$ respectively. By definition,

$$(27) \quad \nu'_{nk}(j, j') = \frac{\hat{\pi}_{n_1jk11}}{\hat{\pi}_{n_1j'k11}} < \nu'_{ik}(j, j') = \frac{\hat{\pi}_{ijk11}}{\hat{\pi}_{ij'k11}} < \nu'_{mk}(j, j') = \frac{\hat{\pi}_{m_1jk11}}{\hat{\pi}_{m_1j'k11}}.$$

From (27), we have the following inequalities

$$(28) \quad \hat{\pi}_{m_1jk11} \hat{\pi}_{ij'k11} > \hat{\pi}_{m_1j'k11} \hat{\pi}_{ijk11}, \quad \hat{\pi}_{n_1j'k11} \hat{\pi}_{ijk11} > \hat{\pi}_{n_1jk11} \hat{\pi}_{ij'k11} \quad \text{for } i \neq m_1, n_1.$$

Now consider part (a). Suppose Condition 1a holds which implies that (25) and (26) are of opposite signs. Using this fact and (28), we observe that $\hat{\alpha}_{i..} < 0$ for at least one and at most

$(I - 1)$ values of Y_1 , that is, boundary solutions of type $\hat{\pi}_{i++2+} = 0$ occur. Using similar arguments, it can be shown that Condition 1b also implies the same. Again, we have

$$\omega'_k(i, i') = \frac{y_{i+k12}}{y_{i'+k12}} = \frac{\sum_j \hat{\pi}_{ijk11} \hat{\beta}_{.j}}{\sum_j \hat{\pi}_{i'jk11} \hat{\beta}_{.j}},$$

$$(29) \quad \omega'_{mk}(i, i') - \omega'_k(i, i') = \frac{\sum_{j \neq m_2} (\hat{\pi}_{im_2k11} \hat{\pi}_{i'jk11} - \hat{\pi}_{i'm_2k11} \hat{\pi}_{ijk11}) \hat{\beta}_{.j}}{\hat{\pi}_{i'm_2k11} \sum_i \hat{\pi}_{i'jk11} \hat{\beta}_{.j}},$$

$$(30) \quad \omega'_k(i, i') - \omega'_{nk}(i, i') = \frac{\sum_{j \neq n_2} (\hat{\pi}_{i'n_2k11} \hat{\pi}_{ijk11} - \hat{\pi}_{in_2k11} \hat{\pi}_{i'jk11}) \hat{\beta}_{.j}}{\hat{\pi}_{i'n_2k11} \sum_i \hat{\pi}_{i'jk11} \hat{\beta}_{.j}},$$

where m_2 and n_2 are the levels of Y_2 corresponding to $\omega'_{mk}(i, i')$ and $\omega'_{nk}(i, i')$ respectively. By definition,

$$(31) \quad \omega'_{nk}(i, i') = \frac{\hat{\pi}_{in_2k11}}{\hat{\pi}_{i'n_2k11}} < \omega'_{jk}(i, i') = \frac{\hat{\pi}_{ijk11}}{\hat{\pi}_{i'jk11}} < \omega'_{mk}(i, i') = \frac{\hat{\pi}_{im_2k11}}{\hat{\pi}_{i'm_2k11}}.$$

From (31), we have the following inequalities

$$(32) \quad \hat{\pi}_{m_2jk11} \hat{\pi}_{ij'k11} > \hat{\pi}_{m_2j'k11} \hat{\pi}_{ijk11}, \quad \hat{\pi}_{n_2jk11} \hat{\pi}_{ij'k11} > \hat{\pi}_{n_2j'k11} \hat{\pi}_{ijk11} \text{ for } j \neq m_2, n_2.$$

Next consider part (b). If Condition 2a holds, then it implies that (29) and (30) are of opposite signs. Using this fact and (32), we observe that $\hat{\beta}_{.j} < 0$ for at least one and at most $(J - 1)$ values of Y_2 , that is, boundary solutions of type $\hat{\pi}_{+j++2} = 0$ occur. Using similar arguments, it can be shown that Condition 2b also implies the same.

Finally, consider part (c). The cases when only one of the condition sets (1a,1b) and (2a,2b) holds have been discussed above. Suppose now both condition sets (1a,1b) and (2a,2b) hold. Then as observed earlier, both $\hat{\alpha}_{i..} < 0$ for at least one and at most $(I - 1)$ values of Y_1 , and $\hat{\beta}_{.j} < 0$ for at least one and at most $(J - 1)$ values of Y_2 , that is, boundary solutions of types $\hat{\pi}_{i++2+} = 0$ and $\hat{\pi}_{+j++2} = 0$ occur. This completes the proof. \square

(iii) $I \times J \times K \times 2 \times 2 \times 2$ table.

Define the odds $\nu''_{ik}(j, j')$, $\nu''_{nk}(j, j')$, $\nu''_{mk}(j, j')$, $\nu''_k(j, j')$, $\nu''_{ij}(k, k')$, $\nu''_{nj}(k, k')$, $\nu''_{mj}(k, k')$ and $\nu''_j(k, k')$ similarly as in (12) and (13). In this case, replace $\hat{\pi}_{ijk1}$ by $\hat{\pi}_{ijk111}$, $\hat{\pi}_{ij'k1}$ by $\hat{\pi}_{ij'k111}$, $\hat{\pi}_{ijk'1}$ by $\hat{\pi}_{ijk'111}$, y_{+jk2} by y_{+jk211} , $y_{+j'k2}$ by $y_{+j'k211}$ and $y_{+jk'2}$ by $y_{+jk'211}$.

Also, define the odds $\omega''_{jk}(i, i')$, $\omega''_{nk}(i, i')$, $\omega''_{mk}(i, i')$, $\omega''_k(i, i')$, $\omega''_{ji}(k, k')$, $\omega''_{ni}(k, k')$, $\omega''_{mi}(k, k')$ and $\omega''_i(k, k')$ similarly as in (19) and (20). In this case, replace $\hat{\pi}_{ijk1}$ by $\hat{\pi}_{ijk111}$, $\hat{\pi}_{ij'k1}$ by $\hat{\pi}_{ij'k111}$, $\hat{\pi}_{ijk'1}$ by $\hat{\pi}_{ijk'111}$, y_{i+k2} by y_{i+k121} , $y_{i'+k2}$ by $y_{i'+k121}$ and $y_{i+k'2}$ by $y_{i+k'121}$.

Finally, define the odds $\delta''_{kj}(i, i')$, $\delta''_{nj}(i, i')$, $\delta''_{mj}(i, i')$, $\delta''_j(i, i')$, $\delta''_{ki}(j, j')$, $\delta''_{ni}(j, j')$, $\delta''_{mi}(j, j')$ and $\delta''_i(j, j')$ similarly as in (21) and (22). In this case, replace $\hat{\pi}_{ijk1}$ by $\hat{\pi}_{ijk111}$, $\hat{\pi}_{ij'k1}$ by $\hat{\pi}_{ij'k111}$, $\hat{\pi}_{ijk'1}$ by $\hat{\pi}_{ijk'111}$, y_{ij+2} by y_{ij+112} , $y_{i'j+2}$ by $y_{i'j+112}$ and $y_{ij'+2}$ by $y_{ij'+112}$.

Here, Y_1 , Y_2 and Y_3 are missing. The following theorem gives sufficient conditions for the occurrence of boundary solutions in this case.

Theorem 4.3. For an $I \times J \times K \times 2 \times 2 \times 2$ incomplete table, consider the following sets (1a, 1b), (2a, 2b) and (3a, 3b) of conditions:

- 1a. $\nu''_k(j, j') \notin (\nu''_{nk}(j, j'), \nu''_{mk}(j, j'))$ for at least one pair (j, j') of Y_2 ,
- 1b. $\nu''_j(k, k') \notin (\nu''_{nj}(k, k'), \nu''_{mj}(k, k'))$ for at least one pair (k, k') of Y_3 .
- 2a. $\omega''_k(i, i') \notin (\omega''_{nk}(i, i'), \omega''_{mk}(i, i'))$ for at least one pair (i, i') of Y_1 ,
- 2b. $\omega''_i(k, k') \notin (\omega''_{ni}(k, k'), \omega''_{mi}(k, k'))$ for at least one pair (k, k') of Y_3 .
- 3a. $\delta''_j(i, i') \notin (\delta''_{nj}(i, i'), \delta''_{mj}(i, i'))$ for at least one pair (i, i') of Y_1 ,
- 3b. $\delta''_i(j, j') \notin (\delta''_{ni}(j, j'), \delta''_{mi}(j, j'))$ for at least one pair (j, j') of Y_2 .

Then we have the following:

- (a) Boundary solutions in NMAR models for only Y_1 , NMAR models for only Y_2 and NMAR models for only Y_3 occur if the sets (1a, 1b), (2a, 2b) and (3a, 3b) of conditions respectively hold.
- (b) Boundary solutions in NMAR models for other combinations of Y_1 , Y_2 and Y_3 occur if at least one of the corresponding combinations of sets (1a,1b), (2a,2b) and (3a,3b) of conditions holds.

Proof. The MLE's $\hat{\alpha}_{i..}$, $\hat{\beta}_{.j}$ and $\hat{\gamma}_{..k}$ under the NMAR model for Y_1 , Y_2 and Y_3 satisfy

$$(33) \quad \sum_i N \hat{\pi}_{ijk111} \hat{\alpha}_{i..} = y_{+jk211} \quad \forall 1 \leq j \leq J, 1 \leq k \leq K,$$

$$(34) \quad \sum_j N \hat{\pi}_{ijk111} \hat{\beta}_{.j} = y_{i+k121} \quad \forall 1 \leq i \leq I, 1 \leq k \leq K,$$

$$(35) \quad \sum_k N \hat{\pi}_{ijk111} \hat{\gamma}_{..k} = y_{ij+112} \quad \forall 1 \leq i \leq I, 1 \leq j \leq J,$$

where $\hat{\pi}_{ijk111} = y_{ijk111}/N$ is the MLE of π_{ijk111} in case of interior (non-boundary) solutions.

The MLE's under NMAR models for other combinations of Y_1 , Y_2 and Y_3 satisfy corresponding combinations of (33), (34) and (35). Note that boundary solutions under the NMAR model for Y_1 , Y_2 and Y_3 occur if at least one of the following conditions is satisfied:

- (i) $\hat{\alpha}_{i..} \leq 0$ for at least one and at most $(I - 1)$ values of Y_1 ,
- (ii) $\hat{\beta}_{.j} \leq 0$ for at least one and at most $(J - 1)$ values of Y_2 ,
- (iii) $\hat{\gamma}_{..k} \leq 0$ for at least one and at most $(K - 1)$ values of Y_3 .

Boundary solutions under NMAR models for other combinations of Y_1 , Y_2 and Y_3 satisfy corresponding combinations of conditions (i), (ii) and (iii). Now

$$(36) \quad \nu''_k(j, j') = \frac{y_{+jk211}}{y_{+j'k211}} = \frac{\sum_i \hat{\pi}_{ijk111} \hat{\alpha}_{i..}}{\sum_i \hat{\pi}_{ij'k111} \hat{\alpha}_{i..}},$$

$$\nu''_{mk}(j, j') - \nu''_k(j, j') = \frac{\sum_{i \neq m_1} (\hat{\pi}_{m_1jk111} \hat{\pi}_{ij'k111} - \hat{\pi}_{m_1j'k111} \hat{\pi}_{ijk111}) \hat{\alpha}_{i..}}{\hat{\pi}_{m_1j'k111} \sum_i \hat{\pi}_{ij'k111} \hat{\alpha}_{i..}},$$

$$(37) \quad \nu''_k(j, j') - \nu''_{nk}(j, j') = \frac{\sum_{i \neq n_1} (\hat{\pi}_{n_1j'k111} \hat{\pi}_{ijk111} - \hat{\pi}_{n_1jk111} \hat{\pi}_{ij'k111}) \hat{\alpha}_{i..}}{\hat{\pi}_{n_1j'k111} \sum_i \hat{\pi}_{ij'k111} \hat{\alpha}_{i..}},$$

where m_1 and n_1 are the levels of Y_1 corresponding to $\nu''_{mk}(j, j')$ and $\nu''_{nk}(j, j')$ respectively. By definition,

$$(38) \quad \nu''_{nk}(j, j') = \frac{\hat{\pi}_{n_1jk111}}{\hat{\pi}_{n_1j'k111}} < \nu''_{ik}(j, j') = \frac{\hat{\pi}_{ijk11}}{\hat{\pi}_{ij'k111}} < \nu''_{mk}(j, j') = \frac{\hat{\pi}_{m_1jk111}}{\hat{\pi}_{m_1j'k111}}.$$

From (38), we have the following inequalities

$$(39) \quad \hat{\pi}_{m_1jk111}\hat{\pi}_{ij'k111} > \hat{\pi}_{m_1j'k111}\hat{\pi}_{ijk111}, \quad \hat{\pi}_{n_1j'k111}\hat{\pi}_{ijk111} > \hat{\pi}_{n_1jk111}\hat{\pi}_{ij'k111} \text{ for } i \neq m_1, n_1.$$

Consider part (a). Condition 1a implies that (36) and (37) are of opposite signs. Using this fact and (39), we observe that $\hat{\alpha}_{i..} < 0$ for at least one and at most $(I - 1)$ values of Y_1 , that is, boundary solutions of type $\hat{\pi}_{i++2++} = 0$ occur. Using similar arguments, it can be shown that Condition 1b also implies the same. Again, we have

$$\omega''_k(i, i') = \frac{y_{i+k121}}{y_{i'+k121}} = \frac{\sum_j \hat{\pi}_{ijk111}\hat{\beta}_{.j}}{\sum_j \hat{\pi}_{i'jk111}\hat{\beta}_{.j}},$$

$$(40) \quad \omega''_{mk}(i, i') - \omega''_{nk}(i, i') = \frac{\sum_{j \neq m_2} (\hat{\pi}_{im_2k111}\hat{\pi}_{i'jk111} - \hat{\pi}_{i'm_2k111}\hat{\pi}_{ijk111})\hat{\beta}_{.j}}{\hat{\pi}_{i'm_2k111} \sum_i \hat{\pi}_{i'jk111}\hat{\beta}_{.j}},$$

$$(41) \quad \omega''_k(i, i') - \omega''_{nk}(i, i') = \frac{\sum_{j \neq n_2} (\hat{\pi}_{i'n_2k111}\hat{\pi}_{ijk111} - \hat{\pi}_{in_2k111}\hat{\pi}_{i'jk111})\hat{\beta}_{.j}}{\hat{\pi}_{i'n_2k111} \sum_i \hat{\pi}_{i'jk111}\hat{\beta}_{.j}},$$

where m_2 and n_2 are the levels of Y_2 corresponding to $\omega''_{mk}(i, i')$ and $\omega''_{nk}(i, i')$ respectively. By definition,

$$(42) \quad \omega''_{nk}(i, i') = \frac{\hat{\pi}_{in_2k111}}{\hat{\pi}_{i'n_2k111}} < \omega''_{jk}(i, i') = \frac{\hat{\pi}_{ijk111}}{\hat{\pi}_{i'jk111}} < \omega''_{mk}(i, i') = \frac{\hat{\pi}_{im_2k111}}{\hat{\pi}_{i'm_2k111}}.$$

From (42), we have the following inequalities

$$(43) \quad \hat{\pi}_{m_2jk111}\hat{\pi}_{ij'k111} > \hat{\pi}_{m_2j'k111}\hat{\pi}_{ijk111}, \quad \hat{\pi}_{n_2j'k111}\hat{\pi}_{ijk111} > \hat{\pi}_{n_2jk111}\hat{\pi}_{ij'k111} \text{ for } j \neq m_2, n_2.$$

Consider now Condition 2a which implies that (40) and (41) are of opposite signs. Using this fact and (43), we observe that $\hat{\beta}_{.j} < 0$ for at least one and at most $(J - 1)$ values of Y_2 , that is, boundary solutions of type $\hat{\pi}_{++j++2+} = 0$ occur. Using similar arguments, it can be shown that Condition 2b also implies the same. Finally we have

$$\delta''_j(i, i') = \frac{y_{ij+112}}{y_{i'j+112}} = \frac{\sum_k \hat{\pi}_{ijk111}\hat{\gamma}_{..k}}{\sum_k \hat{\pi}_{i'jk111}\hat{\gamma}_{..k}},$$

$$(44) \quad \delta''_{mj}(i, i') - \delta''_{nj}(i, i') = \frac{\sum_{k \neq m_3} (\hat{\pi}_{ijm_3111}\hat{\pi}_{i'jk111} - \hat{\pi}_{i'jm_3111}\hat{\pi}_{ijk111})\hat{\gamma}_{..k}}{\hat{\pi}_{i'jm_3111} \sum_k \hat{\pi}_{i'jk111}\hat{\gamma}_{..k}},$$

$$(45) \quad \delta''_j(i, i') - \delta''_{nj}(i, i') = \frac{\sum_{k \neq n_3} (\hat{\pi}_{ijn_3111}\hat{\pi}_{i'jk111} - \hat{\pi}_{i'jn_3111}\hat{\pi}_{ijk111})\hat{\gamma}_{..k}}{\hat{\pi}_{i'jn_3111} \sum_k \hat{\pi}_{i'jk111}\hat{\gamma}_{..k}},$$

where m_3 and n_3 are the levels of Y_3 corresponding to $\delta''_{mj}(i, i')$ and $\delta''_{nj}(i, i')$ respectively. By definition,

$$(46) \quad \delta''_{nj}(i, i') = \frac{\hat{\pi}_{ijn_3111}}{\hat{\pi}_{i'jn_3111}} < \delta''_{kj}(i, i') = \frac{\hat{\pi}_{ijk11}}{\hat{\pi}_{i'jk111}} < \delta''_{mj}(i, i') = \frac{\hat{\pi}_{ijm_3111}}{\hat{\pi}_{i'jm_3111}}.$$

From (45), we have the following inequalities

$$(47) \quad \hat{\pi}_{ijm_3111}\hat{\pi}_{i'jk111} > \hat{\pi}_{i'jm_3111}\hat{\pi}_{ijk111}, \quad \hat{\pi}_{i'jn_3111}\hat{\pi}_{ijk111} > \hat{\pi}_{ijn_3111}\hat{\pi}_{i'jk111} \text{ for } k \neq m_3, n_3.$$

Consider now Condition 3a which implies that (44) and (45) are of opposite signs. Using this fact and (47), we observe that $\hat{\gamma}_{..k} < 0$ for at least one and at most $(K - 1)$ values of Y_3 , that is, boundary solutions of type $\hat{\pi}_{++k++2} = 0$ occur. Using similar arguments, it can be shown that Condition 3b also implies the same.

Next consider part (b). The cases where only one of the condition sets (1a,1b), (2a,2b) and (3a,3b) holds have been discussed above. Assume now other combinations of the condition sets hold. Then as observed earlier, corresponding combinations of conditions (i), (ii) and (iii) are satisfied, that is, boundary solutions of the types $\hat{\pi}_{i++2++} = 0$ or $\hat{\pi}_{+j+++2} = 0$ or $\hat{\pi}_{++k++2} = 0$ (excluding single forms) occur. This completes the proof. \square

Remark 4.1. Note that the response odds $\nu'_{ik}(j, j')$, $\nu'_{ij}(k, k')$, $\omega'_{jk}(i, i')$, and $\omega'_{ji}(k, k')$ in Theorem 4.2 depend only on the fully observed counts $\{y_{ijk11}\}$ for perfect fit models. However, as mentioned earlier, if the missing mechanism is MCAR for at least one of the variables, then perfect fit solutions don't occur. For example, $\hat{\pi}_{ijk11} = \frac{y_{ijk11}y_{++++11}y_{i+k1+}}{Ny_{++++1+}y_{i+k11}}$ in the model $(\alpha_{i..}, \beta_{...})$ (NMAR for Y_1 , MCAR for Y_2) and $\hat{\pi}_{ijk11} = \frac{y_{ijk11}y_{++++11}y_{+jk+1}}{Ny_{++++1+}y_{+jk11}}$ in the model $(\alpha_{...}, \beta_{.j.})$ (NMAR for Y_2 , MCAR for Y_1) (see Section 2). Note that in both the above cases, the response odds $\nu'_{ik}(j, j')$, $\nu'_{ij}(k, k')$, $\omega'_{jk}(i, i')$, and $\omega'_{ji}(k, k')$ in Theorem 4.2 still depend only on the fully observed counts $\{y_{ijk11}\}$. So, the fits of the NMAR models don't matter for calculating the response odds. Also, the partially observed counts are directly used to compute the nonresponse odds $\nu'_k(j, j')$, $\nu'_j(k, k')$, $\omega'_k(i, i')$, and $\omega'_i(k, k')$ in Theorem 4.2. Thus, the fully and partially observed counts in the incomplete tables are sufficient for checking the occurrence of boundary solutions in such models.

5. N-DIMENSIONAL INCOMPLETE TABLE

In this section, we extend the discussions and results in the previous sections to n -dimensional incomplete tables.

5.1. Log-linear parametrization. Let Y_1, \dots, Y_n be n categorical variables with I_1, \dots, I_n levels respectively. Suppose data on k of these variables are missing and data on the remaining $(n - k)$ variables are always observed, where $1 \leq k \leq n$. For $1 \leq i \leq k$, let R_i denote the missing indicator for Y_i such that $R_i = 1$ if data on Y_i is observed and $R_i = 2$ otherwise. Accordingly, we have a variety of incomplete tables, ranging from the $I_1 \times I_2 \times 2$ table (where one variable is missing) to the $I_1 \times \dots \times I_n \times 2^n$ table (where all n variables are missing). There are $\binom{n}{k}$ ways in which k variables may be missing. Without loss of generality, we assume Y_1, \dots, Y_k are missing. Then we have an $I_1 \times \dots \times I_n \times 2^k$ table. The vector of observed counts

is

$$\mathbf{y}_{\text{obs}} = (\{y_{i_1 \dots i_n 1 \dots 1}\}, \{y_{i_1 + \dots + i_{k+1} \dots i_n 12 \dots 21 \dots 1}\}, \dots, \{y_{+ \dots + i_k i_{k+1} \dots i_n 2 \dots 211 \dots 1}\}, \dots, \{y_{+ \dots + i_{k-1} i_k i_{k+1} \dots i_n 2 \dots 2111 \dots 1}\}, \dots, \{y_{i_1 \dots i_{k-1} + i_{k+1} \dots i_n 1 \dots 121 \dots 1}\}, y_{+ \dots + 2 \dots 2}).$$

Note that there are a total of $\prod_{k=1}^n I_k$ fully observed counts and $(2^k - 1)$ supplementary margins. Let $\mu_{i_1 \dots i_n r_1 \dots r_k}$ denote the expected cell frequency, that is, $\mu_{i_1 \dots i_n r_1 \dots r_k} = E(Y_{i_1 \dots i_n r_1 \dots r_k})$. Then the log-linear model is given by

(48)

$$\log \mu_{i_1 \dots i_n r_1 \dots r_k} = \lambda + \sum_{p=1}^n \lambda_{Y_p}(i_p) + \sum_{p \neq q=1}^n \lambda_{Y_p Y_q}(i_p, i_q) + \sum_{p=1}^n \sum_{q=1}^k \lambda_{Y_p R_q}(i_p, r_q) + \sum_{p \neq q=1}^k \lambda_{R_p R_q}(r_p, r_q),$$

where $1 \leq i_l \leq I_l$, $1 \leq l \leq n$, $r_j = 1, 2$, $1 \leq j \leq k$.

Three-way and higher order associations are assumed to be zero in (48) as they are difficult to interpret. It is also assumed that the missingness mechanism of a variable depends on its realization (NMAR) or on any one of the other variables (MAR) or none (MCAR). Note that association terms among Y_i 's and those among R_i 's do not play a role for studying the missing data mechanisms of Y_i 's in (48). Also the missingness mechanism of a variable cannot both be NMAR and MAR simultaneously. So terms involving Y_i, R_i and R_j for $i \neq j$ are excluded in (48). The following constraints are required for identifiability of (48) :

$$\begin{aligned} \sum_{i_p} \lambda_{Y_p}(i_p) &= \sum_{i_p} \lambda_{Y_p Y_q}(i_p, i_q) = \sum_{i_q} \lambda_{Y_p Y_q}(i_p, i_q) = \sum_{i_p} \lambda_{Y_p R_q}(i_p, r_q) = \sum_{r_q} \lambda_{Y_p R_q}(i_p, r_q) \\ &= \sum_{r_p} \lambda_{R_p R_q}(r_p, r_q) = \sum_{r_q} \lambda_{R_p R_q}(r_p, r_q) = 0, \quad p \neq q. \end{aligned}$$

Next, we introduce some parameters to study the missingness mechanisms of Y_1, \dots, Y_k . Define

$$\phi_{i_1 \dots i_n}^p = \frac{P(R_1 = 1, \dots, R_p = 2, \dots, R_{k=1} | Y_1 = i_1, \dots, Y_n = i_n)}{P(R_1 = 1, \dots, R_p = 1, \dots, R_{k=1} | Y_1 = i_1, \dots, Y_n = i_n)}, \quad 1 \leq p \leq k,$$

which describes the missing data mechanism of Y_p . It is the odds of Y_p being missing when the other Y_i 's are observed. There are k such odds. For $i \neq j \neq p = 1, \dots, k$, define

$$\begin{aligned} \theta_{ij} &= \frac{P(R_i = 1, R_j = 1, \{R_p = 1\} | Y_1 = i_1, \dots, Y_n = i_n)}{P(R_i = 1, R_j = 2, \{R_p = 1\} | Y_1 = i_1, \dots, Y_n = i_n)} \\ &\quad \times \frac{P(R_i = 2, R_j = 2, \{R_p = 1\} | Y_1 = i_1, \dots, Y_n = i_n)}{P(R_i = 2, R_j = 1, \{R_p = 1\} | Y_1 = i_1, \dots, Y_n = i_n)}, \end{aligned}$$

which is the conditional odds ratio between R_i and R_j . If $\theta_{ij} = 1$, then the missingness patterns of Y_i and Y_j , that is, R_i and R_j are conditionally independent given that the rest of Y_p 's are observed. There are $\binom{k}{2}$ such ratios. Let $A \subseteq \bar{k} = \{1, \dots, k\}$ such that $|A| \geq 3$. There are $(2^k - (k+1) - \binom{k}{2})$ such sets. Let $R_A = \{R_i | i \in A\}$. Then $\{R_A = 1\} = \{R_i = 1 | i \in A\}$ and $\{R_{\bar{k} \setminus A} = 1\} = \{R_i = 1 | i \notin A\}$. Also, let $2_A = \{r_i = 2 | i \in A\}$, $1_A = \{r_i = 1 | i \in A\}$, $1_{\bar{k} \setminus A} = \{r_i = 1 | i \notin A\}$, $2_{\bar{k} \setminus A} = \{r_i = 2 | i \notin A\}$, $Y_A = \{Y_i | i \in A\}$ and $Y_{\bar{k} \setminus A} = \{Y_i | i \notin A\}$. Now define

$$\theta_A = \frac{P(\{R_A = 2\}, \{R_{\bar{k} \setminus A} = 1\} | Y_1 = i_1, \dots, Y_n = i_n)}{P(\{R_A = 1\}, \{R_{\bar{k} \setminus A} = 1\} | Y_1 = i_1, \dots, Y_n = i_n)} = \frac{\pi_{i_1 \dots i_n 2_A 1_{\bar{k} \setminus A}}}{\pi_{i_1 \dots i_n 1_A 1_{\bar{k} \setminus A}}} = \frac{\mu_{i_1 \dots i_n 2_A 1_{\bar{k} \setminus A}}}{\mu_{i_1 \dots i_n 1_A 1_{\bar{k} \setminus A}}},$$

which is the conditional odds of Y_A being missing given that $Y_{\bar{k} \setminus A}$ are observed. Let $m_{i_1 \dots i_n 1 \dots 1} = \mu_{i_1 \dots i_n 1 \dots 1}$. Then for $1 \leq p \leq k$ and $R_p = 2, \{R_{\bar{k} \setminus \{p\}} = 1\}$, we have $\mu_{i_1 \dots i_n 1 \dots 2 \dots 1} = \phi_{i_1 \dots i_n}^p m_{i_1 \dots i_n 1 \dots 1}$. Also, $m_{i_1 \dots i_n 1 \dots 1} \phi_{i_1 \dots i_n}^r \phi_{i_1 \dots i_n}^s \theta_{rs} = \mu_{i_1 \dots i_n 1_{\{r,s\}} 1_{\bar{k} \setminus \{r,s\}}}$ for $r \neq s = 1, \dots, k$ and $m_{i_1 \dots i_n 1 \dots 1} \theta_A = \mu_{i_1 \dots i_n 2_A 1_{\bar{k} \setminus A}}$. Note that the joint probability

$$\pi_{i_1 \dots i_n + \dots +} = m_{i_1 \dots i_n 1 \dots 1} (1 + \sum_{p=1}^k \phi_{i_1 \dots i_n}^p + \sum_{r \neq s=1}^k \phi_{i_1 \dots i_n}^r \phi_{i_1 \dots i_n}^s \theta_{rs} + \{\theta_A | A \subseteq \bar{k}, |A| \geq 3\}) / N,$$

from which the marginals can be obtained. The total count N is obtained by summing both sides of the above equation over i_1, \dots, i_n . Under (48), the parameters are given as follows.

$$\begin{aligned} \phi_{i_1 \dots i_n}^t &= \exp \left[-2 \left\{ \lambda_{R_t}(1) + \sum_{p=1}^n \lambda_{Y_p R_t}(i_p, 1) + \sum_{p \neq t=1}^k \lambda_{R_p R_t}(1, 1) \right\} \right], \quad 1 \leq t \leq k, \\ \theta_{ij} &= \exp [4 \lambda_{R_i R_j}(1, 1)], \quad i \neq j = 1, \dots, k, \\ \theta_A &= \exp \left[-2 \left\{ \sum_{p=1}^k \lambda_{R_p}(1) + \sum_{p=1}^n \sum_{q=1}^k \lambda_{Y_p R_q}(i_p, 1) \right\} \right], \quad A \subseteq \bar{k}, |A| \geq 3. \end{aligned}$$

We next define the various missing data mechanisms of a variable under (48).

Definition 5.1. If $\phi_{i_1 \dots i_n}^p$ under (48) depends on i_p (denoted by $\phi_{\dots i_p \dots}^p$), then we have a NMAR missingness mechanism for Y_p . If it depends on i_q for $p \neq q$ (denoted by $\phi_{\dots i_q \dots}^p$), then the missingness mechanism for Y_p is MAR, while if it depends on none of i_1, \dots, i_n (denoted by ϕ_{\dots}^p), then the missingness mechanism for Y_p is MCAR.

Since there are $(n+1)$ possible realizations of $\phi_{i_1 \dots i_n}^p$ for each $p = 1, \dots, k$, we have a total of $(n+1)^k$ possible models which may be categorized as follows :

1. MCAR model - the missingness mechanism of each of Y_1, \dots, Y_k is constant (1 case),
2. NMAR model - the missingness mechanism of each of Y_1, \dots, Y_k depends only on itself (1 case),
3. MAR model - the missingness mechanism of each of Y_1, \dots, Y_k depends on any one of the remaining $(n-1)$ variables $((n-1)^k$ cases),
4. Mixture of MCAR and NMAR models - the missingness mechanism of each of Y_1, \dots, Y_k may be MCAR or NMAR, but all variables cannot have the same mechanism $((2^k - 2)$ cases),
5. Mixture of MCAR and MAR models - the missingness mechanism of each of Y_1, \dots, Y_k may be MCAR or MAR, but all variables cannot have the same mechanism $((n^k - (n-1)^k - 1)$ cases),
6. Mixture of NMAR and MAR models - the missingness mechanism of each of Y_1, \dots, Y_k may be NMAR or MAR, but all variables cannot have the same mechanism $((n^k - (n-1)^k - 1)$ cases),
7. Mixture of NMAR, MAR and MCAR models - the missingness mechanism of each of Y_1, \dots, Y_k may be NMAR or MAR or MCAR, but all variables cannot have the same mechanism $((n+1)^k + (n-1)^k - 2(n^k - 1) - 2^k)$ cases).

Note that perfect fits for fully observed counts belong to categories 2, 3 and 6 of models. The log-likelihood kernel under Poisson sampling is

$$\begin{aligned}
l(\mu; \mathbf{y}_{\text{obs}}) &= \sum_{i_1, \dots, i_n} y_{i_1 \dots i_n 1 \dots 1} \log \mu_{i_1 \dots i_n 1 \dots 1} + \sum_{i_2, \dots, i_n} y_{+i_2 \dots i_n 21 \dots 1} \log \mu_{+i_2 \dots i_n 21 \dots 1} \\
&+ \sum_{i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n} y_{i_1 \dots i_{k-1} + i_{k+1} \dots i_n 1 \dots 121 \dots 1} \log \mu_{i_1 \dots i_{k-1} + i_{k+1} \dots i_n 1 \dots 121 \dots 1} + \dots \\
(49) \quad &+ \sum_{i_{k+1}, \dots, i_n} y_{+ \dots + i_{k+1} \dots i_n 2 \dots 21 \dots 1} \log \mu_{+ \dots + i_{k+1} \dots i_n 2 \dots 21 \dots 1} - \sum_{i_1, \dots, i_n, r_1, \dots, r_k} \mu_{i_1 \dots i_n r_1 \dots r_k}.
\end{aligned}$$

Rewriting (49) in terms of the parameters ϕ 's and θ 's, we can obtain closed-form MLE's of the parameters and the expected cell counts under the models described above.

Consider the following hypotheses - H_0 : the proposed model (among models in categories 1 to 7 mentioned above) fits the data, and H_1 : the perfect fit model fits the data. Let L_0 and L_1 denote the maximized log-likelihood functions under the proposed and perfect fit models respectively. Then the likelihood ratio statistic for testing H_0 against H_1 is

$$\begin{aligned}
G^2 &= -2(L_0 - L_1) \\
&= -2 \left[\sum_{i_1, \dots, i_n} \ln \left(\frac{\hat{m}_{i_1 \dots i_n 1 \dots 1}}{y_{i_1 \dots i_n 1 \dots 1}} \right) + \sum_{i_2, \dots, i_n} y_{+i_2 \dots i_n 21 \dots 1} \ln \left(\frac{\sum_{i_1} \hat{m}_{i_1 \dots i_n 1 \dots 1} \hat{\phi}_{i_1 \dots i_n}^1}{y_{+i_2 \dots i_n 21 \dots 1}} \right) \right. \\
&\quad + \dots + \sum_{i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n} y_{i_1 \dots i_{k-1} + i_{k+1} \dots i_n 1 \dots 121 \dots 1} \ln \left(\frac{\sum_{i_k} \hat{m}_{i_1 \dots i_n 1 \dots 1} \hat{\phi}_{i_1 \dots i_n}^k}{y_{i_1 \dots i_{k-1} + i_{k+1} \dots i_n 1 \dots 121 \dots 1}} \right) \\
&\quad + \dots + \sum_{i_{k+1}, \dots, i_n} y_{+ \dots + i_{k+1} \dots i_n 2 \dots 21 \dots 1} \ln \left(\frac{\hat{m}_{+ \dots + i_{k+1} \dots i_n 1 \dots 1} \hat{\theta}_{1 \dots k}}{y_{+ \dots + i_{k+1} \dots i_n 2 \dots 21 \dots 1}} \right) \\
(50) \quad &\left. - \sum_{i_1, \dots, i_n} \hat{m}_{i_1 \dots i_n 1 \dots 1} \left(1 + \sum_{p=1}^k \hat{\phi}_{i_1 \dots i_n}^p + \sum_{r \neq s=1}^k \hat{\phi}_{i_1 \dots i_n}^r \hat{\phi}_{i_1 \dots i_n}^s \hat{\theta}_{rs} + \{\hat{\theta}_A | A \subseteq \bar{k}, |A| \geq 3\} \right) + N \right].
\end{aligned}$$

Note that $G^2 \sim \chi_\nu^2$ asymptotically, where $\nu = (\prod_{p=k+1}^n I_p) \prod_{r \neq p=1}^k (1 + I_r)$ - Number of free estimable parameters under the proposed model.

5.2. Boundary solutions. Boundary solutions occur if the MLE's of any of the parameters ϕ 's < 0 , which are then set to zero to obtain boundary estimates. Note that for $1 \leq p \leq k$, we have $\hat{\phi}_{\dots i_p \dots}^p = 0$ for at least one and at most $(I_p - 1)$ values of Y_p in case of boundary solutions. Equivalently, the boundary solutions in an $I_1 \times \dots \times I_n \times 2^k$ table, where Y_1, \dots, Y_k are missing, take at least one of the following forms.

1. $\hat{\pi}_{i_1 + \dots + 12 \dots 2} = 0$ for at least one and at most $(I_1 - 1)$ values of Y_1 ,
2. $\hat{\pi}_{+i_2 + \dots + 212 \dots 2} = 0$ for at least one and at most $(I_2 - 1)$ values of Y_2 ,
- \vdots
- k. $\hat{\pi}_{+ \dots + i_k + \dots + 2 \dots 212 \dots 2} = 0$ for at least one and at most $(I_k - 1)$ values of Y_k .

Next, we establish a sufficient condition for the occurrence of boundary solutions in NMAR models of types 2, 4, 6 and 7. For $1 \leq p \leq k$, $1 \leq q \leq n$, $p \neq q$ and any pair (i_q, i'_q) of Y_q , let

$S_q = \{i_1, \dots, i_n\} \setminus \{i_q\}$ and $S_{pq} = \{i_1, \dots, i_n\} \setminus \{i_p, i_q\}$. Define the response odds

$$\begin{aligned}\tau_{S_q}^p(i_q, i'_q) &= \frac{\hat{\pi}_{\{i_q\} \cup S_q \cup \{1 \dots 1\}}}{\hat{\pi}_{\{i'_q\} \cup S_q \cup \{1 \dots 1\}}}, \quad \tau_{\{n_p^*\} \cup S_{pq}}^p(i_q, i'_q) = \min_{i_p} \left\{ \tau_{S_q}^p(i_q, i'_q) \right\}, \\ \tau_{\{m_p^*\} \cup S_{pq}}^p(i_q, i'_q) &= \max_{i_p} \left\{ \tau_{S_q}^p(i_q, i'_q) \right\}\end{aligned}$$

and the nonresponse odds

$$\tau_{S_{pq}}^p(i_q, i'_q) = \frac{y_{i_1 \dots i_{p-1} + i_{p+1} \dots i_q \dots i_n 1 \dots 1 2 1 \dots 1 \dots 1}}{y_{i_1 \dots i_{p-1} + i_{p+1} \dots i'_q \dots i_n 1 \dots 1 2 1 \dots 1 \dots 1}},$$

where n_p^* and m_p^* denote the levels of Y_p for which $\tau_{S_q}^p(i_q, i'_q)$ is minimum and maximum respectively. Then the following theorem provides a sufficient condition for the occurrence of boundary solutions in a n -dimensional incomplete table.

Theorem 5.1. For an $I_1 \times \dots \times I_n \times 2^k$ incomplete contingency table, consider the following k sets of conditions:

1. $\tau_{S_{1q}}^1(i_q, i'_q) \notin \left(\tau_{\{n_1^*\} \cup S_{1q}}^1(i_q, i'_q), \tau_{\{m_1^*\} \cup S_{1q}}^1(i_q, i'_q) \right)$ for at least one pair (i_q, i'_q) of Y_q , where $q = 2, \dots, n$,
2. $\tau_{S_{2q}}^2(i_q, i'_q) \notin \left(\tau_{\{n_2^*\} \cup S_{2q}}^2(i_q, i'_q), \tau_{\{m_2^*\} \cup S_{2q}}^2(i_q, i'_q) \right)$ for at least one pair (i_q, i'_q) of Y_q , where $q = 1, 3, \dots, n$,
- \vdots
- k. $\tau_{S_{kq}}^k(i_q, i'_q) \notin \left(\tau_{\{n_k^*\} \cup S_{kq}}^k(i_q, i'_q), \tau_{\{m_k^*\} \cup S_{kq}}^k(i_q, i'_q) \right)$ for at least one pair (i_q, i'_q) of Y_q , where $q = 1, \dots, k-1, k+1, n$.

Then we have the following:

- (a) Boundary solutions in NMAR models for only Y_1 , for only Y_2 and so on for only Y_k occur if the sets 1, 2, \dots , k of conditions respectively hold.
- (b) Boundary solutions in NMAR models for other combinations of Y_1, \dots, Y_k occur if at least one of the corresponding combinations of sets 1, 2, \dots , k of conditions holds.

Proof. The proof is similar to that of Theorem 4.3. □

6. DATA ANALYSIS

In this section, we illustrate our results in Sections 2 and 4 using a real-life example from Rubin *et al.* (1995). Consider first the results in Section 2. The Slovenian public opinion (SPO) survey dataset given in Table 4 below is a $2 \times 2 \times 2 \times 2 \times 2$ table classified by the variables Secession (Y_1), Attendance (Y_2) and Independence (Y_3), each having two levels Yes (1) and No (2). The “Don’t know” category (missing margins) for each variable is denoted by “Missing”. We replace the count 0 by 2 in the full table. The total cell count is 2076, among which data on all three variables are observed ($R_1 = R_2 = R_3 = 1$) for 1456 persons, Y_1 and Y_2 observed ($R_1 = R_2 = 1, R_3 = 2$) for 57 persons, Y_1 and Y_3 observed ($R_1 = R_3 = 1, R_2 = 2$) for 171 persons, Y_2 and Y_3 observed ($R_2 = R_3 = 1, R_1 = 2$) for 95 persons, only Y_1 observed ($R_2 = R_3 = 2, R_1 = 1$) for 40 persons, only Y_2 observed ($R_1 = R_3 = 2, R_2 = 1$) for 134 persons, only Y_3 observed ($R_1 = R_2 = 2, R_3 = 1$) for 27 persons, and all missing ($R_1 = R_2 = R_3 = 2$) for 96 persons.

Table 4. Data from the SPO survey.

			$R_3 = 1$ $Y_3 = 1$	$Y_3 = 2$	$R_3 = 2$ Missing	
$R_1 = 1$	$Y_1 = 1$	$R_2 = 1$	$Y_2 = 1$	1191	8	21
			$Y_2 = 2$	8	2	4
		$R_2 = 2$	Missing	107	3	9
	$Y_1 = 2$	$R_2 = 1$	$Y_2 = 1$	158	68	29
			$Y_2 = 2$	7	14	3
		$R_2 = 2$	Missing	18	43	31
$R_1 = 2$	Missing	$R_2 = 1$	$Y_2 = 1$	90	2	109
			$Y_2 = 2$	1	2	25
		$R_2 = 2$	Missing	19	8	96

Without loss of generality, consider the subtable of Table 4 in which data on only Y_1 is missing as given below.

Table 5. Subtable Y_1 of Table 4.

			$Y_3 = 1$ $Y_3 = 2$
$R = 1$	$Y_1 = 1$	$Y_2 = 1$	1191 8
		$Y_2 = 2$	8 2
	$Y_1 = 2$	$Y_2 = 1$	158 68
		$Y_2 = 2$	7 14
$R = 2$	Missing	$Y_2 = 1$	90 2
		$Y_2 = 2$	1 2

To determine the missing data mechanism, we fit models 1 to 4 (see Subsection 2.1) to the data in Table 5. The system of equations for model 2 (NMAR for Y_1) yields $\hat{\alpha}_{1..} = 0.0721$ and $\hat{\alpha}_{2..} = 0.0258$ implying that boundary solutions do not occur. We use the ‘ecm.cat’ function of the ‘cat’ package in R software to fit the above models using the EM algorithm. Let G^2 denote the likelihood ratio statistic for testing the goodness of fit of each of the models 1 to 4 against the model in (1). The table below gives the log-likelihoods, G^2 values and degrees of freedom (d.f.) for the tests.

Table 6. Comparison of fit among models.

Model	Boundary solution	Log-likelihood	G^2	d.f.
$\alpha_{i..}$	No	-1382.45	1.12	2
$\alpha_{.j.}$	No	-1382.55	1.33	2
$\alpha_{..k}$	No	-1382.37	0.96	2
$\alpha_{...}$	No	-2100.48	1437.18	3

From the above table, we deduce that the best fit model is $\alpha_{..k}$ (MAR for Y_1) based on minimum G^2 value = 0.96 and p -value = 0.6177. This implies that the missingness in the variable ‘Secession’ depends on the observed variable ‘Independence’. This dependence is expected because if one is unsure about voting for Slovenian’s secession from Yugoslavia, then one is also most likely decided about Slovenian independence. Note that ‘Secession’ differs from ‘Independence’ here since independence without secession was also possible with the

formation of a new internal state. The table of expected cell counts using the closed-form estimates (see Subsection 2.1) is given below.

Table 7. Expected cell counts for model $\alpha_{.,k}$ using closed-form estimates.

			$Y_3 = 1$	$Y_3 = 2$
$R = 1$	$Y_1 = 1$	$Y_2 = 1$	1191.00	7.87
		$Y_2 = 2$	8.00	2.16
	$Y_1 = 2$	$Y_2 = 1$	158.00	66.93
		$Y_2 = 2$	7.00	15.09
$R = 2$	$Y_1 = 1$	$Y_2 = 1$	79.46	0.34
		$Y_2 = 2$	0.53	0.09
	$Y_1 = 2$	$Y_2 = 1$	10.54	2.91
		$Y_2 = 2$	0.47	0.66

Next, consider without loss of generality the subtable of Table 4 in which data on Y_1 and Y_2 are missing as given below.

Table 8. Subtable $Y_1 Y_2$ of Table 4.

			$Y_3 = 1$	$Y_3 = 2$
$R_1 = 1$	$Y_1 = 1$	$R_2 = 1 \quad Y_2 = 1$	1191	8
		$Y_2 = 2$	8	2
		$R_2 = 2 \quad \text{Missing}$	107	3
	$Y_1 = 2$	$R_2 = 1 \quad Y_2 = 1$	158	68
		$Y_2 = 2$	7	14
		$R_2 = 2 \quad \text{Missing}$	18	43
$R_1 = 2$	Missing	$R_2 = 1 \quad Y_2 = 1$	90	2
		$Y_2 = 2$	1	2
		$R_2 = 2 \quad \text{Missing}$	19	8

To determine the missing data mechanism, we fit models 1 to 16 (see Subsection 2.2) to the data in Table 8. On solving the systems of equations in NMAR models for Y_1 or Y_2 , we obtain $\hat{\alpha}_{1..} = 0.0721$, $\hat{\alpha}_{2..} = 0.0258$, $\hat{\beta}_{1.} = 0.073$ and $\hat{\beta}_{2.} = 2.375$. Hence, there are no boundary solutions. We use the ‘ecm.cat’ function of the ‘cat’ package in R software to fit the above models using the EM algorithm. Let G^2 denote the likelihood ratio statistic for testing the goodness of fit of each of the models 1 to 16 against the model in (4). The table below gives the log-likelihoods, G^2 values and degrees of freedom (d.f.) for the tests.

Table 9. Comparison of fit among models.

Model	Boundary solution	Log-likelihood	G^2	d.f.
$(\alpha_{...}, \beta_{...})$	No	-2294.26	74.30	6
$(\alpha_{...}, \beta_{i..})$	No	-2275.51	36.78	5
$(\alpha_{...}, \beta_{.j.})$	No	-2258.46	2.68	5
$(\alpha_{...}, \beta_{..k})$	No	-2258.55	2.87	5
$(\alpha_{i..}, \beta_{...})$	No	-2294.12	74.01	5
$(\alpha_{i..}, \beta_{i..})$	No	-2275.50	36.78	4
$(\alpha_{i..}, \beta_{.j.})$	No	-2258.41	2.58	4
$(\alpha_{i..}, \beta_{..k})$	No	-2258.48	2.74	4
$(\alpha_{.j.}, \beta_{...})$	No	-2293.71	73.19	5
$(\alpha_{.j.}, \beta_{i..})$	No	-2275.06	35.89	4
$(\alpha_{.j.}, \beta_{.j.})$	No	-2258.20	2.18	4
$(\alpha_{.j.}, \beta_{..k})$	No	-2258.32	2.42	5
$(\alpha_{..k}, \beta_{...})$	No	-2294.134	74.04	5
$(\alpha_{..k}, \beta_{i..})$	No	-2275.49	36.76	4
$(\alpha_{..k}, \beta_{.j.})$	No	-2258.38	2.54	4
$(\alpha_{..k}, \beta_{..k})$	No	-2258.47	2.71	4

From the above table, we deduce that the best fit model is $(\alpha_{.j.}, \beta_{.j.})$ (MAR for Y_1 , NMAR for Y_2) based on minimum G^2 value = 2.18 and p -value = 0.7031. This implies that the missingness in each of the variables ‘Secession’ and ‘Attendance’ depends on the variable ‘Attendance’. This is due to the fact that if one is unsure about attending the plebiscite, then data on ‘Secession’ will also be missing most probably and vice-versa. Similarly, data on ‘Attendance’ will be missing. The table of expected cell counts using the closed-form estimates (see Subsection 2.2) is given below.

Table 10. Expected cell counts under model $(\alpha_{.j.}, \beta_{.j.})$ using closed-form estimates.

			$Y_3 = 1$	$Y_3 = 2$
$R_1 = 1$	$Y_1 = 1$	$R_2 = 1 \quad Y_2 = 1$	1191.00	8.00
		$Y_2 = 2$	8.00	2.00
		$R_2 = 2 \quad Y_2 = 1$	86.94	0.58
		$Y_2 = 2$	19.00	4.75
	$Y_1 = 2$	$R_2 = 1 \quad Y_2 = 1$	158.00	68.00
		$Y_2 = 2$	7.00	14.00
		$R_2 = 2 \quad Y_2 = 1$	11.53	4.96
		$Y_2 = 2$	16.62	33.25
$R_1 = 2$	$Y_1 = 1$	$R_2 = 1 \quad Y_2 = 1$	76.89	0.52
		$Y_2 = 2$	0.77	0.19
		$R_2 = 2 \quad Y_2 = 1$	10.95	0.07
		$Y_2 = 2$	3.59	0.90
	$Y_1 = 2$	$R_2 = 1 \quad Y_2 = 1$	10.20	4.39
		$Y_2 = 2$	0.68	1.35
		$R_2 = 2 \quad Y_2 = 1$	1.45	0.62
		$Y_2 = 2$	3.14	6.28

Note that $\hat{\theta} = 1.9507$ for the model $(\alpha_{.j}, \beta_{.j})$, which implies that the missing mechanisms of the variables ‘Secession’ and ‘Attendance’ are not independent as observed earlier. The conditional probability of Y_1 being missing given $Y_2 = 1$ is observed is $\hat{\phi}_{1|2}(1) = \frac{\hat{\alpha}_{1.}}{1+\hat{\alpha}_{1.}} = 0.0606$. Similarly, the conditional probability of Y_1 being missing given $Y_2 = 2$ is observed is $\hat{\phi}_{1|2}(2) = \frac{\hat{\alpha}_{2.}}{1+\hat{\alpha}_{2.}} = 0.0882$. So the probability of nonresponse for ‘Secession’ is greater when one replies ‘No’ to attending the plebiscite. Also, the conditional probability of Y_2 being missing given $Y_1 = 1$ is observed is $\hat{\phi}_{2|1}(1) = \frac{\hat{\beta}_{.1}}{1+\hat{\beta}_{.1}} = 0.068$. Similarly, the conditional probability of Y_2 being missing given $Y_1 = 2$ is observed is $\hat{\phi}_{2|1}(2) = \frac{\hat{\beta}_{.2}}{1+\hat{\beta}_{.2}} = 0.7037$. Hence, the probability of nonresponse for ‘Attendance’ is greater when one replies ‘No’ to Slovenia’s secession from Yugoslavia. From Remark 2.1 and the data in Table 8, we have $OR_{..1} = OR_{111} = 6.5957$ and $OR_{..2} = OR_{112} = 0.8235$ for the model $(\alpha_{.j}, \beta_{.j})$. This implies that if none of the responses for the variables is missing, then the estimated odds ratio between ‘Secession’ and ‘Attendance’ is greater when the response to ‘Independence’ is ‘No’ than when it is ‘Yes’. Also, $Var(OR_{..1}) = 11.9646$ and $Var(OR_{..2}) = 0.4823$, that is, for observed data, the estimated odds ratio between ‘Secession’ and ‘Attendance’ has greater precision when the response to ‘Independence’ is ‘No’ than when it is ‘Yes’.

Now, we illustrate our results in Section 4. To investigate the occurrence of boundary solutions, we consider subtables of Table 4 in which at least one of Y_1 , Y_2 and Y_3 is missing. When we fit perfect fit NMAR models (for fully observed counts) to the data in each subtable, we observe that boundary solutions do not occur in any of them as the MLE’s are $\hat{\alpha}_{1.} = 0.0721$, $\hat{\alpha}_{2.} = 0.0258$, $\hat{\beta}_{.1} = 0.073$, $\hat{\beta}_{.2} = 2.375$, $\hat{\gamma}_{..1} = 0.0151$ and $\hat{\gamma}_{..2} = 0.3851$. According to Theorems 4.1 to 4.3, this implies at least one of the sufficient conditions in each case should be violated which is indeed true for the subtables. Also, we modify some fully observed counts in each subtable so that the sufficient conditions in Theorems 4.1 to 4.3 are satisfied. Table 11 shows the MLE’s for perfect fit (for fully observed counts) NMAR models in the modified subtables.

Table 11. MLE's in modified subtables of Table 4.

Subtable	Changes	NMAR models (perfect fit)	MLE's	Boundary solns.
Y_1	$158 \rightarrow 1300, 68 \rightarrow 28,$ $7 \rightarrow 10$	Y_1	$\hat{\alpha}_{1..} = -0.0293, \hat{\alpha}_{2..} = 0.0961$	$\hat{\pi}_{1++2} = 0$
Y_2	$8 \rightarrow 80$	Y_2	$\hat{\beta}_{.1.} = -0.1258, \hat{\beta}_{.2.} = 3.2391$	$\hat{\pi}_{+1+2} = 0$
Y_3	$8 \rightarrow 55, 14 \rightarrow 6$	Y_3	$\hat{\gamma}_{.1.} = -0.0024, \hat{\gamma}_{.2.} = 0.4338$	$\hat{\pi}_{++12} = 0$
Y_1Y_2	$158 \rightarrow 3100, 7 \rightarrow 10,$ $2 \rightarrow 1$ $8 \rightarrow 80$ $158 \rightarrow 3100, 7 \rightarrow 10,$ $2 \rightarrow 1$	Y_1 Y_2 Y_1, Y_2	$\hat{\alpha}_{1..} = -0.0123, \hat{\alpha}_{2..} = 0.0338$ $\hat{\beta}_{.1.} = -0.1258, \hat{\beta}_{.2.} = 3.2391$ $\hat{\alpha}_{1..} = -0.0123, \hat{\alpha}_{2..} = 0.0338$ $\hat{\beta}_{.1.} = -0.0004, \hat{\beta}_{.2.} = 4.5899$	$\hat{\pi}_{1++2+} = 0$ $\hat{\pi}_{+1++2} = 0$ $\hat{\pi}_{1++2+} = 0$ $\hat{\pi}_{+1++2} = 0$
Y_1Y_3	$158 \rightarrow 1100, 68 \rightarrow 22,$ $7 \rightarrow 10$ $8 \rightarrow 55, 14 \rightarrow 6$ $1191 \rightarrow 3191, 8 \rightarrow 28,$ $8 \rightarrow 28, 2 \rightarrow 5,$ $158 \rightarrow 1100, 68 \rightarrow 20,$ $7 \rightarrow 10, 14 \rightarrow 2$	Y_1 Y_3 Y_1, Y_3	$\hat{\alpha}_{1..} = -0.0346, \hat{\alpha}_{2..} = 0.1193$ $\hat{\gamma}_{.1.} = -0.0024, \hat{\gamma}_{.2.} = 0.4338$ $\hat{\alpha}_{1..} = -0.0139, \hat{\alpha}_{2..} = 0.1222$ $\hat{\gamma}_{.1.} = -0.0094, \hat{\gamma}_{.2.} = 1.8458$	$\hat{\pi}_{1++2+} = 0$ $\hat{\pi}_{++1+2} = 0$ $\hat{\pi}_{1++2+} = 0$ $\hat{\pi}_{++1+2} = 0$
Y_2Y_3	$8 \rightarrow 80, 14 \rightarrow 10$ $8 \rightarrow 80, 14 \rightarrow 6$ $8 \rightarrow 108, 8 \rightarrow 108,$ $2 \rightarrow 4, 14 \rightarrow 2$	Y_2 Y_3 Y_2, Y_3	$\hat{\beta}_{.1.} = -0.1937, \hat{\beta}_{.2.} = 4.2616$ $\hat{\gamma}_{.1.} = -0.0132, \hat{\gamma}_{.2.} = 0.4588$ $\hat{\beta}_{.1.} = 0.2138, \hat{\beta}_{.2.} = -1.3706$ $\hat{\gamma}_{.1.} = -0.0253, \hat{\gamma}_{.2.} = 0.4785$	$\hat{\pi}_{+1+2+} = 0$ $\hat{\pi}_{++1+2} = 0$ $\hat{\pi}_{+1+2+} = 0$ $\hat{\pi}_{++1+2} = 0$
$Y_1Y_2Y_3$	$158 \rightarrow 1100, 68 \rightarrow 22,$ $7 \rightarrow 10$ $8 \rightarrow 80$ $8 \rightarrow 55, 14 \rightarrow 6$ $158 \rightarrow 3100, 7 \rightarrow 10$ $1191 \rightarrow 3191, 8 \rightarrow 48,$ $8 \rightarrow 28, 2 \rightarrow 4,$ $158 \rightarrow 1000, 68 \rightarrow 20,$ $7 \rightarrow 10, 14 \rightarrow 2$ $8 \rightarrow 55, 8 \rightarrow 80,$ $2 \rightarrow 4, 14 \rightarrow 2$	Y_1 Y_2 Y_3 Y_1, Y_2 Y_1, Y_3 Y_2, Y_3	$\hat{\alpha}_{1..} = -0.0346, \hat{\alpha}_{2..} = 0.1193$ $\hat{\beta}_{.1.} = -0.1258, \hat{\beta}_{.2.} = 3.2391$ $\hat{\gamma}_{.1.} = -0.0024, \hat{\gamma}_{.2.} = 0.4338$ $\hat{\alpha}_{1..} = -0.0081, \hat{\alpha}_{2..} = 0.0322$ $\hat{\beta}_{.1.} = -0.0002, \hat{\beta}_{.2.} = 4.5387$ $\hat{\alpha}_{1..} = -0.0291, \hat{\alpha}_{2..} = 0.1828$ $\hat{\gamma}_{.1.} = -0.0397, \hat{\gamma}_{.2.} = 3.1164$ $\hat{\beta}_{.1.} = 0.4620, \hat{\beta}_{.2.} = -5.5516$ $\hat{\gamma}_{.1.} = -0.0022, \hat{\gamma}_{.2.} = 0.4326$	$\hat{\pi}_{1++2++} = 0$ $\hat{\pi}_{+1++2+} = 0$ $\hat{\pi}_{++1++2} = 0$ $\hat{\pi}_{+1++2+} = 0$ $\hat{\pi}_{+1++2+} = 0$ $\hat{\pi}_{+1++2+} = 0$ $\hat{\pi}_{++1++2} = 0$ $\hat{\pi}_{+1++2+} = 0$

From the above table, we observe that after fitting perfect fit NMAR models to the modified subtables, boundary solutions occur in each of them since at least one of $\hat{\alpha}_{i..}$, $\hat{\beta}_{.j.}$ and $\hat{\gamma}_{.k.}$ is negative. The same results hold on fitting non-perfect fit NMAR models to the various subtables. In the last column of Table 11, the boundary solutions under all the above models are obtained using the EM algorithm (see the 'ecm.cat' function of the 'cat' package in R software). The forms of boundary solutions under the various models are the same as those mentioned in Subsection 4.1.

The next example illustrates the occurrence of boundary solutions in one of the subtables.

Example 6.1. Consider the subtable below in which data on only Y_1 is missing.

Table 12. Subtable Y_1 of Table 4.

			$Y_3 = 1$	$Y_3 = 2$
$R_1 = 1$	$Y_1 = 1$	$Y_2 = 1$	1191	8
		$Y_2 = 2$	8	2
	$Y_1 = 2$	$Y_2 = 1$	158	68
		$Y_2 = 2$	7	14
$R_1 = 2$	Missing	$Y_2 = 1$	90	2
		$Y_2 = 2$	1	2

If we fit a perfect fit NMAR model for Y_1 , then there are no boundary solutions as $\hat{\alpha}_{1..}$ and $\hat{\alpha}_{2..} > 0$. From Theorem 4.1, this implies at least one of the sufficient conditions is violated, which is true since $\frac{90}{2} \in (\frac{158}{68}, \frac{1191}{8})$ and $\frac{90}{1} \in (\frac{158}{7}, \frac{1191}{8})$. If we make the changes $158 \rightarrow 1300$, $68 \rightarrow 28$ and $7 \rightarrow 10$ in Table 3, then the sufficient conditions in Theorem 4.1 hold since $\frac{90}{2} \notin (\frac{1300}{28}, \frac{1191}{8})$, $\frac{1}{2} \notin (\frac{10}{14}, \frac{8}{2})$, $\frac{90}{1} \notin (\frac{1300}{10}, \frac{1191}{8})$ and $\frac{2}{2} \notin (\frac{28}{14}, \frac{8}{2})$. On fitting the same model now, we observe from Table 11 that $\hat{\alpha}_{1..} < 0$, which implies boundary solutions occur.

7. CONCLUSIONS

In this paper, we have proposed sufficient conditions for the occurrence of boundary solutions under various NMAR models for $I \times J \times K \times 2$, $I \times J \times K \times 2 \times 2$ and $I \times J \times K \times 2 \times 2 \times 2$ incomplete contingency tables. These conditions depend only on observed counts and hence are easily verifiable. To study missing data mechanisms in the above tables, we have considered hierarchical log-linear models which yield closed-form MLE's of parameters and expected cell counts. Note that the methods and results in this paper are applicable for $I \times J \times 2$ and $I \times J \times 2 \times 2$ tables also. Finally, extensions of the models and results on boundary solutions are presented for arbitrary n -dimensional incomplete tables. A real-life data analysis example validates our modelling approach and other results in this paper.

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